Probability Theory
4. Almost sure convergence and convergence in probability

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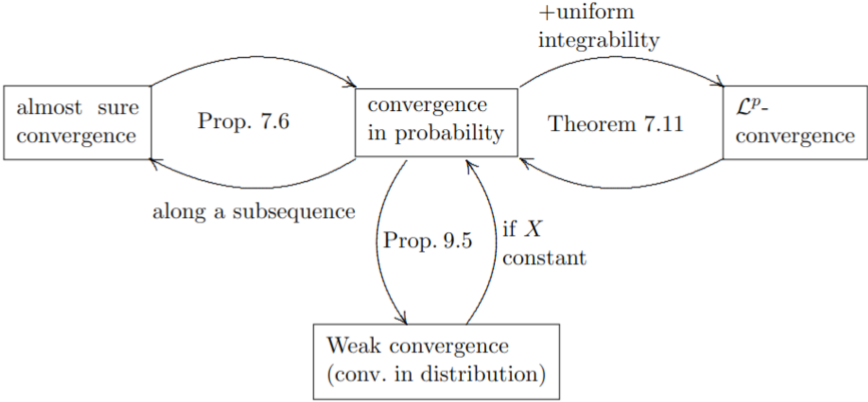
Kinds of convergence



Characterization of convergence in probability

- Lemma 7.5: $X, X_1, X_2, \ldots$ RVs with values in (E, r).

$$X_n \xrightarrow{n\to\infty}_p X \qquad \Longleftrightarrow \qquad \mathrm{E}[r(X_n, X) \wedge 1] \xrightarrow{n\to\infty} 0.$$

Proof: $\Rightarrow \forall \varepsilon > 0:$

$$\lim_{n\to\infty} \mathrm{E}[r(X_n, X) \wedge 1] \leq \lim_{n\to\infty} \big(\varepsilon + \mathrm{P}(r(X_n, X) > \varepsilon)\big) = \varepsilon.$$

$\Leftarrow$ It follows with the Chebyshev inequality for $0 < \varepsilon \leq 1$ that

$$\mathrm{P}(r(X_n, X) > \varepsilon) \leq \frac{\mathrm{E}[r(X_n, X) \wedge 1]}{\varepsilon} \xrightarrow{n\to\infty} 0.$$

Convergence in probability and almost sure convergence

- Proposition 7.6: $X, X_1, X_2, \ldots$ RVs with values in (E, r).

  Then, the following are equivalent:

  1. $X_n \xrightarrow{n\to\infty}_p X$
  2. For every $(n_k)_{k=1,2,\ldots}$ there is a subsequence $(n_{k_\ell})_{\ell=1,2,\ldots}$ with $X_{n_{k_\ell}} \xrightarrow{\ell\to\infty}_{fs} X$.

  1. $\to$ 2.: We can use a subsequence $(n_{k_\ell})_{\ell=1,2,\ldots}$ so that

$$E\Big[\sum_{\ell=1}^{\infty}(r(X_{n_{k_\ell}}, X) \wedge 1)\Big] = \sum_{\ell=1}^{\infty} E[r(X_{n_{k_\ell}}, X) \wedge 1] < \infty,$$

$$1 = P\Big(\sum_{\ell=1}^{\infty}(r(X_{n_{k_\ell}}, X) \wedge 1) < \infty\Big) \le P\big(\limsup_{\ell\to\infty} r(X_{n_{k_\ell}}, X) = 0\big) \le 1.$$

Convergence in probability and almost sure convergence

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  Then, the following are equivalent:

  1. $X_n \xrightarrow{n\to\infty}_p X$
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  We show $(\neg 1. \wedge 2.) \Rightarrow$ Contradiction. With n_k and $\varepsilon > 0$ from $\neg 1.$, $(n_{k_\ell})_\ell$ from 2:

$$\varepsilon < \liminf_{k\to\infty} \mathrm{E}[r(X_{n_k}, X) \wedge 1] \leq \lim_{\ell\to\infty} \mathrm{E}[r(X_{n_{k_\ell}}, X) \wedge 1] = 0.$$