Probability Theory
3. Different kinds of convergence

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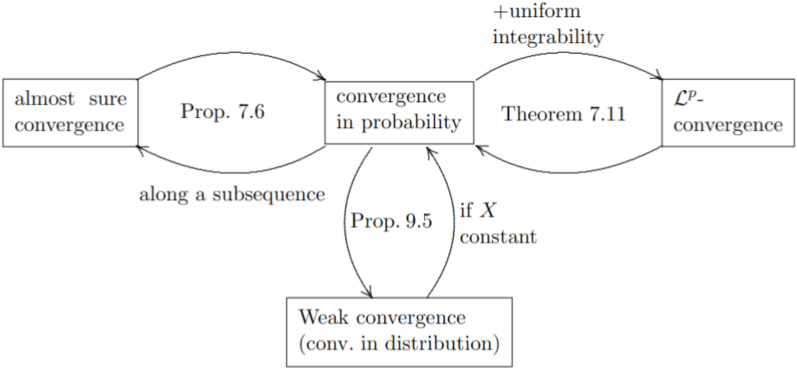
Kinds of convergence



Definition of kinds of convergence

- Definition 7.1: $X, X_1, X_2, \ldots$ RVs with values in (E, r).
  - $X_1, X_2, \ldots$ converges almost surely to X if
$$\mathbf{P}(\lim_{n\to\infty} r(X_n, X) = 0) = 1. \qquad X_n \xrightarrow{n\to\infty}_{fs} X$$
  - $X_1, X_2, \ldots$ converges stochastically to X if
$$\forall \varepsilon > 0 : \lim_{n\to\infty} \mathbf{P}(r(X_n, X) > \varepsilon) = 0. \qquad X_n \xrightarrow{n\to\infty}_{p} X.$$
  - $E = \mathbb{R}$; $X_1, X_2, \ldots$ converges to $\mathcal{L}^p$ against X if
$$\lim_{n\to\infty} \mathbf{E}[|X_n - X|^p] = 0. \qquad X_n \xrightarrow{n\to\infty}_{\mathcal{L}^p} X$$

$\mathcal{L}^p$-convergence

- For example, if $X, X_1, X_2, \dots$ such that $X_n \xrightarrow{n\to\infty}_{\mathcal{L}^q} X$ and $p < q$, then holds

  $$\mathbf{E}[|X_n - X|^p] = \mathbf{E}[(|X_n - X|^q)^{p/q}] \leq \mathbf{E}[|X_n - X|^q]^{p/q} \xrightarrow{n\to\infty} 0,$$

  thus $X_n \xrightarrow{n\to\infty}_{\mathcal{L}^p} X$.
- $\mathcal{L}^p$ is complete, so: If there are therefore for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $m, n \geq n$

  $$\mathbf{E}[|X_n - X_m|^p] < \varepsilon,$$

  then there is a random variable $X \in \mathcal{L}^p$ with $X_n \xrightarrow{n\to\infty}_{\mathcal{L}^p} X$.

counterexamples

Let $U \sim U([0,1])$.

- $(X_n \xrightarrow{n\to\infty}_p X) \not\Rightarrow (X_n \xrightarrow{n\to\infty}_{fs} X)$ with $X = 0$ and

  $X_n := 1_{U \in A_n}$,

  $A_1 = [0, \frac{1}{2}]; A_2 = [\frac{1}{2}, 1]; A_3 = [0, \frac{1}{4}]; A_4 = [\frac{1}{4}, \frac{2}{4}]; A_5 = [\frac{2}{4}, \frac{3}{4}]; A_6 = [\frac{3}{4}, 1$

  Then applies $\lim_{n\to\infty} \mathbf{P}(|X_n| > \varepsilon) = \lim_{n\to\infty} \mathbf{P}(U \in A_n) = 0$,

  i.e. $X_n \xrightarrow{n\to\infty}_p 0$, but for each $n \in \mathbb{N}$ there is an $m > n$ with

  $X_m = 1$. Therefore $X_n \not\xrightarrow{n\to\infty}_{fs} 0$

- $(Y_n \xrightarrow{n\to\infty}_{fs} Y) \not\Rightarrow (Y_n \xrightarrow{n\to\infty}_{\mathcal{L}^p} Y)$

  with $Y = 0$ and $Y_n := n \cdot 1_{U \in B_n}$ for $B_n = [0, \frac{1}{n}]$. The

  following applies

  $$\mathbf{P}(\lim_{n\to\infty} X_n = 0) = \mathbf{P}(U > 0) = 1,$$

Stochastic limit is unique

- Lemma 8.4: $X, Y, X_1, X_2, \dots$ ZV with values in (E, r), $X_n \xrightarrow{n\to\infty}_p X$ and $X_n \xrightarrow{n\to\infty}_p Y$. Then $X = Y$ is almost certain.
- proof:

$$\begin{aligned}\mathbf{P}(X \neq Y) &= \mathbf{P}\Big(\bigcup_{k=1}^{\infty} \big\{r(X,Y) > 2/k\big\}\Big) \leq \sum_{k=1}^{\infty} \mathbf{P}(r(X,Y) > 2/k) \\ &\leq \sum_{k=1}^{\infty} \limsup_{n\to\infty} (\mathbf{P}(r(X,X_n) > 1/k) + \mathbf{P}(r(X,Y_n) > 1/k)) \\ &= 0.\end{aligned}$$