universität freiburg

Probability Theory

Summer semester 2024

Lecture: Prof. Dr. Peter Pfaffelhuber
Assistance: Samuel Adeosun
https://pfaffelh.github.io/hp/2024ss_wtheorie.html
https://www.stochastik.uni-freiburg.de/

Tutorial 4 - Almost sure, stochastic and $\mathcal{L}^p$-convergence

Exercise 1 (1+1+1+1+1+1 Points)**.**
Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$ and $\mathbf{P} = \lambda|_{[0,1]}$ be the Lebesgue measure on $[0,1]$. The sequence $(X_n)_{n\in\mathbb{N}}$ of random variables is given by

$$X_1 = 0 \text{ and } X_n = \sqrt{n}\mathbb{1}_{(\frac{1}{n},\frac{2}{n})}.$$

Prove or disprove:

(a) $(X_n)_{n\in\mathbb{N}}$ converges in probability.

(b) $(X_n)_{n\in\mathbb{N}}$ converges almost surely.

(c) $(X_n^2)_{n\in\mathbb{N}}$ converges almost surely.

(d) $(X_n)_{n\in\mathbb{N}}$ converges in $\mathcal{L}^2$.

(e) $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable.

(f) $(X_n^2)_{n\in\mathbb{N}}$ is uniformly integrable.

Exercise 2 (1+1+1+1+1+1 Points)**.**
Let X, Y, X_1, Y_1, X_2, $Y_2, \dots$ be random variables on a probability space $(\Omega,\mathcal{F},\mathbf{P})$ with $X_n \sim Y_n$ for all n and $X \sim Y$. Prove or disprove (by giving a counter-example) the following statements:

(a) $X_n \to_p 0 \iff Y_n \to_p 0$.

(b) $X_n \to_{as} 0 \iff Y_n \to_{as} 0$.

(c) $X_n \to_{\mathcal{L}^1} 0 \iff Y_n \to_{\mathcal{L}^1} 0$.

(d) $X_n \to_p X \iff Y_n \to_p Y$.

(e) $X_n \to_{as} X \iff Y_n \to_{as} Y$.

(f) $X_n \to_{\mathcal{L}^1} X \iff Y_n \to_{\mathcal{L}^1} Y$.

Recall that $X \sim Y$ means that $X_*\mathbf{P} = Y_*\mathbf{P}$.

Exercise 3 (2+2 Points)**.**
Let $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ and $(Z_n)_{n\in\mathbb{N}}$ be sequences of real, integrable random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with $X_n \leq Y_n \leq Z_n$ for all $n \in \mathbb{N}$ and $X_n \to_p X$, $Y_n \to_p Y$ and $Z_n \to_p Z$. Show that:

(a) $X_n + Y_n \to_p X + Y$.

(b) If $\mathbf{E}[X_n] \to \mathbf{E}[X]$, and $\mathbf{E}[Z_n] \to \mathbf{E}[Z]$ also applies, then $\mathbf{E}[Y_n] \to \mathbf{E}[Y]$.

Exercise 4 (2+2 bonus points)**.**
Let $p,q \in \mathbb{N}$ and $X_1, X_2, \ldots \sim \exp(1)$ be independent. In addition, let $Y = X_1 + \cdots + X_p$ and $Z = X_{p+1} + \cdots + X_{p+q}$.

(a) Show that $Y \sim f_p \cdot \lambda$, where λ is Lebesgue measure and

$$f_p(x) = \frac{1}{(p-1)!} x^{p-1} e^{-x} 1_{x\geq 0}.$$

(b) Show that $Y/(Y + Z) \sim \beta(p,q)$, the β-distribution from the last sheet.