universität freiburg

Probability Theory

Summer semester 2024

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https://pfaffelh.github.io/hp/2024ss_wtheorie.html https://www.stochastik.uni-freiburg.de/

Tutorial 1 - Review of measure theory

Exercise 1 (4 Points)**.**
Let Ω be a finite set such that $|\Omega| \geq 4$ and even. Set

$$\mathcal{D} := \{D \subset \Omega \mid |D| \in 2\mathbb{N}\}.$$

Show that $\mathcal{D}$ is a Dynkin system, but not a σ-algebra.

Solution.
Since $|\Omega|$ is even, $\Omega \in \mathcal{D}$ is even. If $A,B \in \mathcal{D}$ with $A \subset B$, then surely $|B \setminus A| = |B| - |A|$ is even and therefore $B \setminus A \in \mathcal{D}$. Finally, let $A_1,A_2,\ldots \in \mathcal{D}$ be an ascending sequence. Then, since $|\Omega| < \infty$, it also holds that $|\bigcup_{k\geq 1} A_k| \leq |\Omega| < \infty$. In particular, there exists n such that $\bigcup_{k\geq 1} A_k = \bigcup_{k=1}^{n} A_k = A_n \in \mathcal{D}$. $\mathcal{D}$ is therefore a Dynkin system according to Definition 1.11.

However, $\mathcal{D}$ cannot be $\cap$-stable and therefore cannot be a σ-algebra (see Table 1). This is because, since $|\mathcal{D}| \geq 2$ we can find three different ω_1, ω_2 ,ω_3 $\in \Omega$. Then $\{\omega_1, \omega_2\} \in \mathcal{D}$ and $\{\omega_2, \omega_3\} \in \mathcal{D}$ but not $\{\omega_1, \omega_2\} \cap \{\omega_2, \omega_3\} = \{\omega_2\}$.

Exercise 2 (1+3=4 Points)**.**
Let μ^* be an outer measure on Ω.

1. Prove that if $\mu^*(A) = 0$, then $\mu^*(A \cup B) = \mu^*(B)$.
2. Let (Ω,r) be a metric space, and μ^* the outer measure from Proposition 2.15, where $\mathcal{F}$ is the topology generated from (Ω,r). In addition, let A and B be bounded sets for which there is an $\alpha > 0$ such that $r(a,b) \geq \alpha$ for all $a \in A,b \in B$. Prove that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Solution.

1. By the monotonicity of μ^*, we have that:

$$\mu^*(B) \leq \mu^*(A \cup B) \quad (\text{since } \; A \subseteq A \cup B). \tag{1}$$

Also, by $\sigma-$subadditivity of μ^*, we have:

$$\mu^*(A \cup B) \leq \underbrace{\mu^*(A)}_{0} + \mu^*(B). \tag{2}$$

From (1) and (2), we establish that $\mu^*(A \cup B) = \mu^*(B)$.

2. By the $\sigma-$subadditivity of μ^*, we know that

$$\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B).$$

Hence we only need to show that the reverse inequality holds. Now fix $\varepsilon > 0$. Since A and B are bounded, $A \cup B$ is bounded; and $\mu^*(A \cup B)$ is finite. We can therefore find a countable collection of non-empty, open, bounded intervals $\{I_k\}_{k=1}^{\infty}$ which covers $A \cup B$ (such that $A \cup B \subseteq \bigcup_{k=1}^{\infty} I_k$) and satisfies:

$$\mu^*(A \cup B) > \sum_{k=1}^{\infty} l(I_k) - \varepsilon$$

Without loss of generality, assume the length of each interval in the collection is less than $\frac{\alpha}{2}$ (the intervals can be subdivided until this condition holds). Then by construction, each interval only intersect either A or B. Define,

$$\mathcal{A} = \{k : I_k \cap A \neq \emptyset\} \quad \text{and} \quad \mathcal{B} = \{k : I_k \cap B \neq \emptyset\}.$$

Since $\{I_k\}_{k \in \mathcal{A}}$ and $\{I_k\}_{k \in \mathcal{B}}$ form open covers of A and B respectively, we can conclude:

$$\mu^*(A \cup B) > \sum_{k \in \mathcal{A}} l(I_k) + \sum_{k \in \mathcal{B}} l(I_k) - \epsilon \geq \mu^*(A) + \mu^*(B) - \varepsilon.$$

This expression holds for all $\varepsilon > 0$, so we must have $\mu^*(A \cup B) \geq \mu^*(A) + \mu^*(B)$. Therefore, $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Exercise 3 (2+2=4 Points)**.**

1. Let μ^* an outer measure on Ω. Show that if E_1 and E_2 are measurable, then

$$\mu^*(E_1 \cup E_2) + \mu^*(E_1 \cap E_2) = \mu^*(E_1) + \mu^*(E_2).$$

2. Let (Ω,r) is a metric space. Show that if a set $E \subseteq \Omega$ has positive outer measure, then there is a bounded subset of E that also has positive outer measure.

Solution.

1. Since E_2 is measurable, we have

$$\mu^*(E_1 \cup E_2) = \mu^* \underbrace{((E_1 \cup E_2) \cap E_2)}_{E_2} + \mu^* \underbrace{((E_1 \cup E_2) \cap E_2^c)}_{E_1 \setminus E_2}. \tag{3}$$

Again, by the measurablility of E_2,

$$\mu^*(E_1) = \mu^*(E_1 \cap E_2) + \mu^* \underbrace{(E_1 \cap E_2^c)}_{E_1 \setminus E_2}. \tag{4}$$

Combining (3) and (4), we have

$$\mu^*(E_1 \cup E_2) + \mu^*(E_1 \cap E_2) = \mu^*(E_1) + \mu^*(E_2).$$

2. *Approach: Assume that every bounded subset of E has measure zero, then establish that the measure of E is consequently zero.*

   Let $x_0 \in E$ and consider $\mathcal{B}(x_0,r)$, $r \in \mathcal{Q}^+$ balls in Ω. Then $\mathcal{B}(x_0,r) \cap E$ is a bounded subset of E such that $E \subseteq \bigcup_{r\in\mathcal{Q}^+} \mathcal{B}(x_0,r)\cap E$. By monotonicity and $\sigma-$subadditivity of μ^*,

$$\mu^*(E) \leq \mu^* \left(\bigcup_{r\in\mathcal{Q}^+} \mathcal{B}(x_0,r) \cap E \right) \leq \sum_{r\in\mathcal{Q}^+} \underbrace{\mu^* \left(\mathcal{B}(x_0,r) \cap E\right)}_{0}$$

   Hence, $\mu^*(E) = 0$. Thus, if a set $E \subseteq \Omega$ has positive outer measure, then there is a bounded subset of E that also has positive outer measure.

Exercise 4 (1.5+0.5+1.5=4 Points)**.**

1. Prove that the set of all real numbers which do not have a 6 in their decimal representation, is a Lebesgue 0-set.
2. Randomly choose an independent and identically, uniformly distributed sequence of numbers in $\{0,...,9\}$. Compute the probability that the first n numbers are not 6.
3. Do you see a connection between (1) and (2)?

Solution.

1. Let

$$E = \{x \in \mathbb{R} : x \text{ does not have a six in its representaton.}\}$$

   Write

$$\mathbb{R} = \bigcup_{n\in\mathbb{Z}} (n,n+1)$$

   Let

$$\tilde{E}_n = E \cap (n,n+1), \quad n \in \mathbb{Z}$$

   Then,

$$\bigcup_{n\in\mathbb{Z}} \tilde{E}_n = \bigcup_{n\in\mathbb{Z}} E \cap (n,n+1) = E \cap \mathbb{R} = E.$$

   So that,

$$\mu(E) = \mu\left(\bigcup_{n\in\mathbb{Z}} \tilde{E}_n\right) = \sum_{n\in\mathbb{Z}} \mu(\tilde{E}_n)$$

   Now, consider

$$E_0 = E \cap (0,1).$$

   Define some sets F_n such that

$$F_0 = \{\text{the first digit after the decimal is a six}\} = [0.6,0.7), \quad \mu(F_0) = 0.1 = \frac{1}{10}.$$

$$F_1 = \{\text{the second digit after the decimal is a six}\} = [0.06,0.07) \cup [0.16,0.17) \cup \ldots \cup [0.96,0.97).$$

We observe however that the set [0.66,0.067) in F_1 coincides with F_0 and since we want these sets to be disjoint, we can remove the reoccurence in F_1 so that $\mu(F_1) = 0.01 \times 9 = 0.09 = \frac{9}{10^2}$. In the same manner,

$F_2 = \{\text{the third digit after the decimal is a six}\} = [0.006{,}0.007) \cup [0.016{,}0.017) \cup \ldots \cup [0.096{,}0.097)$.

So that $\mu(F_2) = 0.01 \times 9 \times 9 = \frac{9^2}{10^3}$. Then,

$$E_0^c = \bigcup_{n \geq 0} F_n = \bigcup_{n=0}^{\infty} F_n.$$

and

$$\mu\left(\tilde{E}_0^c\right) = \sum_{n=0}^{\infty} \frac{9^n}{10^{n+1}} = \frac{1}{10} \sum_{n=0}^{\infty} \frac{9^n}{10^n} = \lim_{N \to \infty} \frac{1}{10} \sum_{n=0}^{N} \left(\frac{9}{10}\right)^n = \frac{1}{10} \cdot \frac{1}{1 - \frac{9}{10}} = 1$$

Hence,

$$\mu(E_0) = 1 - \mu(\tilde{E}_0^c) = 1 - 1 = 0.$$

In the same way, $\mu(\tilde{E}_n^c) = 0 \quad \forall\, n \in \mathbb{Z}$.

2. To compute the probability that the first n numbers in a randomly chosen independent and identically uniformly distributed sequence of numbers in $\{0, \ldots, 9\}$ are not 6, we need to determine the total number of possible sequences and the number of sequences where the first n numbers are not 6. The total number of possible sequences of length n is given by 10^n because each digit in the sequence can take on one of the ten possible values (0,1,2,…,or 9) independently. Besides, the total number of posible sequences which are not six will be 9^n. Therefore, the number of sequences of length n where the first n numbers are not 6 is $(9^n/10^n) = (9/10)^n$. *We can also think of it this way: let us count the number of sequences where the first n numbers are not 6. Since each digit is chosen independently and uniformly, the probability of any digit being 6 is $1/10$. Therefore, the probability of each digit not being 6 is $1 - 1/10 = 9/10$. Hence, the number of sequences of length n where the first n numbers are not 6 is $(9/10)^n$.*

3. Now to see a connection between (1) and (2), let us compute the probability that the first n numbers are not 6 in a special way. This is similar to generating the decimal part of a real number. Suppose we fix the integer part, that is we consider numbers of the form $X.d_1d_2\ldots$. What we are interested in can be written mathematically as $\text{Prob}(d_1, d_2, \ldots, d_n \text{ such that } d_i \neq 6, \ \forall i = 1, \ldots, n)$. This is equivalent to $\text{Prob}(d_1, d_2, \ldots, d_n \text{ such that } d_i = 6, \text{ for some } i = 1, \ldots, n)$ which will then be equal to

$$1 - P\left(\bigcup_{i=0}^{n-1} F_i\right) = 1 - \mu\left(\bigcup_{i=0}^{n-1} F_i\right) \quad \text{(think of measure as probability)}$$

So that

$$1 - \sum_{i=0}^{n-1} \frac{9^i}{10^{i+1}} = 1 - \frac{1}{10} \sum_{i=0}^{n-1} \frac{9}{10}^{i} = 1 - \frac{1}{10}\left(1.\frac{1 - \left(\frac{9}{10}\right)^n}{1 - \frac{9}{10}}\right) = \left(\frac{9}{10}\right)^n$$