Measure Theory for Probabilists
16. Measures on product spaces

Peter Pfaffelhuber

March 1, 2024

Definition 5.9

$(\Omega_i, \mathcal{F}_i), i = 1, 2$ measurable spaces.

- $\kappa : \Omega_1 \times \mathcal{F}_2 \to \mathbb{R}_+$ is a *transition kernel from* $(\Omega_1, \mathcal{F}_1)$ *to* $(\Omega_2, \mathcal{F}_2)$ if
  (i) for all $\omega_1 \in \Omega_1$, the map $\kappa(\omega_1, .)$ is a measure on $\mathcal{F}_2$ and
  (ii) for all $A_2 \in \mathcal{F}_2$ $\kappa(., A_2)$ is $\mathcal{F}_1$-measurable.
- A transition kernel is called σ-finite if there is a sequence $\Omega_{21}, \Omega_{22}, \dots \in \mathcal{F}_2$ with $\Omega_{2n} \uparrow \Omega_2$ and $\sup_{\omega_1} \kappa(\omega_1, \Omega_{2n}) < \infty$ for all $n = 1, 2, \dots$
- It is called *stochastic kernel* or *Markov kernel* if for all $\omega_1 \in \Omega_1$ the map $\kappa(\omega_1, .)$ is a probability measure.

Example: Markov chain

- $\Omega = \{\omega_1, \dots, \omega_n\}$ finite and $P = (p_{ij})_{1 \le i,j \le n}$ with $p_{ij} \in [0,1]$ and $\sum_{j=1}^n p_{ij} = 1$. Then,

$$\kappa(\omega_i, .) := \sum_{j=1}^{n} p_{ij} \cdot \delta_{\omega_j}$$

  is a Markov kernel from $(\Omega, 2^\Omega)$ to $(\Omega, 2^\Omega)$.
- P is the transition matrix of a homogeneous, Ω-valued Markov chain.

- $(\Omega_i, \mathcal{F}_i), i = 1, 2$ be measurable spaces, μ a σ-finite measure on $\mathcal{F}_0$, κ a σ-finite transition kernel from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$
- Lemma 5.11: Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}_+$ be $\mathcal{F}_1 \otimes \mathcal{F}_2$ measurable. Then,

$$\omega_1 \mapsto \kappa(\omega_1, .)[f] := \int \kappa(\omega_1, d\omega_2) f(\omega_1, \omega_2)$$

  is $\mathcal{F}_1$-measurable.
- Theorem 5.12: There is exactly one σ-finite measure $\mu \otimes \kappa$ on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ with

$$\Big(\mu \otimes \kappa\Big)(A \times B) = \int_A \mu(d\omega_1)\Big(\int_B \kappa(\omega_1, d\omega_2)\Big).$$

Fubini's Theorem

- Theorem 5.13: $(\Omega_i, \mathcal{F}_i)$, μ, κ and $\mu \otimes \kappa$ as above. Let $f : \Omega_1 \to \Omega_2 \to \mathbb{R}_+$ measurable with respect to $\mathcal{F}_1 \otimes \mathcal{F}_2$. Then,

$$\int fd\big(\mu \otimes \kappa\big) = \int \mu(d\omega_1)\Big(\int \kappa(\omega_1, d\omega_2) f(\omega_1, \omega_2)\Big).$$

  Equality also applies if $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable with $\int |f| d\big(\mu \otimes \kappa\big) < \infty$.
- Corollary 5.14: Let $\Omega = \Omega_1 \times \Omega_2$ and $\mathcal{H}_i \subseteq 2^{\Omega_i}$ be a semi-ring, and $\mu_i : \mathcal{H}_i \to \mathbb{R}_+$ σ-finite and, σ-additive, $i = 1, 2$. Then there is exactly one measure $\mu_1 \otimes \mu_2$ on $\sigma(\mathcal{H}_1) \otimes \sigma(\mathcal{H}_2)$ with

$$\mu_1 \otimes \mu_2(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2).$$

  For $f : \Omega \to \mathbb{R}_+$ measurable, the value of the integral does not depend on the order of integration.

Definition and Example

- $\lambda^{\otimes d}$ is d-dimensional Lebesgue measure. Let

$$f(x,y) = \frac{xy}{(x^2+y^2)^2}.$$

Then, for every $x \in \mathbb{R}$

$$\int \lambda(dy) f(x,y) = 0,$$

since $f(x,.) \in \mathcal{L}^1(\lambda)$ and $f(x,y) = -f(x,-y)$. Therefore, iterated integrals are 0. However, $|f|$ is not integrable because f has a non-integrable pole in $(0,0)$.

Convolutions of measures 1

- Definition 5.17: Let μ_1, μ_2 be σ-finite measures on $\mathcal{B}(\mathbb{R})$ and $\mu_1 \otimes \mu_2$ their product measure. Let $S(x_1, x_2) := x_1 + x_2$. Then $S_*(\mu_1 \otimes \cdots \otimes \mu_n)$ is the *convolution* of μ_1, μ_2 and is denoted by $\mu_1 * \mu_2$.
- $\gamma_1, \gamma_2 \geq 0$, $\mu_{\text{Poi}(\gamma_1)}$ and $\mu_{\text{Poi}(\gamma_2)}$. Then,

$$\begin{aligned}
\mu_{\text{Poi}(\gamma_1)} * \mu_{\text{Poi}(\gamma_2)} &= \sum_{m,n} 1_{m+n=k} e^{-(\gamma_1+\gamma_2)} \frac{\gamma_1^m \gamma_2^n}{m!n!} \cdot \delta_k \\
&= \sum_{m=0}^{k} e^{-(\gamma_1+\gamma_2)} \frac{\gamma_1^m \gamma_2^{k-m}}{m!(k-m)} \cdot \delta_k \\
&= e^{-(\gamma_1+\gamma_2)} \frac{(\gamma_1+\gamma_2)^k}{k!} \cdot \delta_k \sum_{m=0}^{k} \binom{k}{m} \frac{\gamma_1^m \gamma_2^{k-m}}{(\gamma_1+\gamma_2)^k} \\
&= \mu_{\text{Poi}(\gamma_1+\gamma_2)}.
\end{aligned}$$

Convolutions of measures 2

- Lemma 5.19: λ measure on $\mathcal{B}(\mathbb{R})$, $\mu = f_\mu \cdot \lambda$ and $\nu = f_\nu \cdot \lambda$. Then,$\mu * \nu = f_{\mu*\nu} \cdot \lambda$ with

$$f_{\mu*\nu}(t) = \int f_\mu(s) f_\nu(t-s)\lambda(ds).$$

- $f_{N(\mu_1,\sigma_1^2)}$ and $f_{N(\mu_2,\sigma_2^2)}$. Let $\mu := \mu_1 + \mu_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$. Then, the density of $N(\mu_1,\sigma_1^2) * N(\mu_2,\sigma_2^2)$ is

$$\begin{aligned}
x \mapsto & \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2}} \int \exp\Big(- \frac{(y-\mu_1)^2}{2\sigma_1^2} - \frac{(x-y-\mu_2)^2}{2\sigma_2^2}\Big) dy \\
& = \dots = \\
& = \frac{1}{2\pi\sigma} \int \exp\Big(- \frac{(\sigma y - \frac{\sigma_1}{\sigma_2}(x-\mu))^2}{2\sigma^2} - \frac{(x-\mu)^2\Big(\frac{\sigma^2}{\sigma_2^2} - \frac{\sigma_1^2}{\sigma_2^2}\Big)}{2\sigma^2}\Big) dy \\
& = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(- \frac{(x-\mu)^2}{2\sigma^2}\Big).
\end{aligned}$$