Measure Theory for Probabilists
14. Theorem of Radon-Nikodým

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Theorem of Radon-Nikodým

- Corollary 4.17: μ, ν be σ-finite measures. Then, ν has a density with respect to μ if and only if $\nu \ll \mu$.

- Theorem 4.16 (Lebesgue decomposition theorem): μ, ν be σ-finite measures. Then ν can be written uniquely as

$$\nu = \nu_a + \nu_s \qquad \text{with} \qquad \nu_a \ll \mu, \nu_s \perp \mu.$$

  The measure ν_a has a density with respect to μ that is μ-almost everywhere finite.

Absolute continuity

- Definition 4.13: ν has a *density* f with respect to μ if for all $A \in \mathcal{F}$,
$$\nu(A) = \mu[f; A].$$
We write $f = \frac{d\nu}{d\mu}$ and $\nu = f \cdot \mu$.
- ν is *absolutely continuous with respect to* μ if all μ-zero sets are also ν-zero sets. We write $\nu \ll \mu$. If both $\nu \ll \mu$ and $\mu \ll \nu$, then μ and ν are called *equivalent*.
- μ and ν are called *singular* if there is an $A \in \mathcal{F}$ with $\mu(A) = 0$ and $\nu(A^c) = 0$. We write $\mu \perp \nu$.

Chain rule

- Lemma 4.14: Let μ be a measure on $\mathcal{F}$.
  1. Let ν be a σ-finite measure. If g_1 and g_2 are densities of ν with respect to μ, then $g_1 = g_2$, μ-almost everywhere.
  2. Let $f : \Omega \to \mathbb{R}_+$ and $g : \Omega \to \mathbb{R}$ be measurable. Then,
  $$(f \cdot \mu)[g] = \mu[fg],$$
  if one of the two sides exists.
- Proof for finite μ: 1. Set $A := \{g_1 > g_2\}$. Since both g_1 and g_2 are densities of ν with respect to μ,
  $$0 = \nu(A) - \nu(A) = \mu[g_1 - g_2; A].$$
  Since only $g_1 > g_2$ is possible on A, $g_1 = g_2$ is $1_A\mu$-almost everywhere.
  2. For $g = 1_A$ with $A \in \mathcal{F}$, write
  $$(f \cdot \mu)[g] = (f \cdot \mu)(A) = \mu[f, A] = \mu[f 1_A] = \mu[fg].$$
  This extends up to the general case.

Examples

- For $\mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}_+$
$$f_{N(\mu,\sigma^2)}(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)$$
and λ is the one-dimensional Lebesgue measure. Then, $f_{N(\mu,\sigma^2)} \cdot \lambda$ is a *normal distribution*.
- For $\gamma \geq 0$, let
$$f_{\exp(\gamma)}(x) := 1_{x\geq 0} \cdot \gamma e^{-\gamma x}.$$
Then, $f_{\exp(\gamma)} \cdot \lambda$ is called *exponential distribution with parameter* γ. From the chain rule,
$$\mathrm{E}[X] = f_{\exp(\gamma)} \cdot \lambda[\mathrm{id}] = \int_0^\infty \gamma e^{-\gamma x} x dx = ... = \frac{1}{\gamma}.$$
- Let μ be the counting measure on $\mathbb{N}_0$ and
$$f(k) = e^{-\gamma}\frac{\gamma^k}{k!}, \qquad k = 0, 1, 2, ...$$
Then $f \cdot \mu$ is the Poisson distribution for the parameter γ.

Theorem 4.16

- Let μ, ν be σ-finite measures. Then ν can be written uniquely as
$$\nu = \nu_a + \nu_s \qquad \text{with} \qquad \nu_a \ll \mu, \nu_s \perp \mu.$$
The measure ν_a has a density with respect to μ that is μ-almost everywhere finite.
- Proof for finite μ, ν. The map
$$\begin{cases} \mathcal{L}^2(\mu+\nu) & \to \mathbb{R}) \\ f & \mapsto \nu[f] \end{cases}$$
is continuous. By Riesz-Frechet, there is $h \in \mathcal{L}^2(\mu+\nu)$ with
$$\nu[f] = (\mu+\nu)[fh], \qquad \nu[f(1-h)] = \mu[fh], \qquad f \in \mathcal{L}^2(\mu+\nu).$$
For $f = 1_{\{h<0\}}$ and $f = 1_{\{h>1\}}$, we find
$$0 \le \nu\{h<0\} = (\mu+\nu)[h; h<0] \le 0,$$
$$0 \le \mu[h; \{h>1\}] = \nu[1-h; \{h>1\} \le 0.$$

Theorem 4.16

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$$\nu = \nu_a + \nu_s \qquad \text{with} \qquad \nu_a \ll \mu, \nu_s \perp \mu.$$

The measure ν_a has a density with respect to μ that is μ-almost everywhere finite.

- Proof: Let $E := h^{-1}\{1\}$, and $f = 1_E$. Then,

$$\mu(E) = \mu[h; E] = \nu[1 - h; E] = 0.$$

Define $\nu = \nu_a + \nu_s$ and $\nu_s \perp \mu$ using

$$\nu_a(A) = \nu(A \setminus E), \qquad \nu_s(A) = \nu(A \cap E),$$

To show: $\nu_a \ll \mu$, so choose $A \in \mathcal{F}$ with $\mu(A) = 0$, so

$$\nu[1 - h; A \setminus E] = \mu[h; A \setminus E] = 0.$$

Since $h < 1$ on $A \setminus E$, $\nu_a(A) = \nu(A \setminus E) = 0$, i.e. $\nu_a \ll \mu$.

Theorem 4.16

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The measure ν_a has a density with respect to μ that is μ-almost everywhere finite.
- Proof: Define $\nu = \nu_a + \nu_s$ and $\nu_s \perp \mu$ using
$$\nu_a(A) = \nu(A \setminus E), \qquad \nu_s(A) = \nu(A \cap E),$$
To show: $g := \frac{h}{1-h} 1_{\Omega \setminus E}$ is the density of ν_a with respect to μ:
$$\mu[g; A] = \mu\Big[\frac{h}{1-h}; A \setminus E\Big] = \nu(A \setminus E) = \nu_a(A).$$
Uniqueness: let $\nu = \nu_a + \nu_s = \widetilde{\nu}_a + \widetilde{\nu}_s$ Choose $A, \widetilde{A} \in \mathcal{A}$ with $\nu_s(A) = \mu(A^c) = \widetilde{\nu}_s(\widetilde{A}) = \mu(\widetilde{A}^c) = 0$. Then,
$$\nu_a = 1_{A \cap \widetilde{A}} \cdot \nu_a = 1_{A \cap \widetilde{A}} \cdot \nu = 1_{A \cap \widetilde{A}} \cdot \widetilde{\nu}_a = \widetilde{\nu}_a.$$

Corollary 4.17

- Let μ, ν be σ-finite measures. Then, ν has a density with respect to μ if and only if $\nu \ll \mu$.
- Proof: '$\Rightarrow$': clear. '$\Leftarrow$': Lebesgue decomposition Theorem, there is a unique decomposition $\nu = \nu_a + \nu_s$ with $\nu_a \ll \mu, \nu_s \perp \mu$. Since $\nu \ll \mu$, $\nu_s = 0$ must apply and therefore $\nu = \nu_a$. In particular, the density of ν exists with respect to μ.