Measure Theory for Probabilists
11. Convergence results

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Outline

- Theorem 3.25 for Riemann integral:
  $f, f_1, f_2, \ldots : [a, b] \to \mathbb{R}$ be piecewise continuous with $f_n \xrightarrow{n\to\infty} f$ uniformly. Then
  $$\int_a^b f_n(x)dx \xrightarrow{n\to\infty} \int_a^b f(x)dx.$$
- Theorem 3.26, monotone convergence:
  $f_1, f_2, \cdots \in \mathcal{L}^1(\mu)$ and $f : \Omega \to \overline{\mathbb{R}}$ measurable with $f_n \uparrow f$ almost everywhere. Then,
  $$\lim_{n\to\infty} \mu[f_n] = \mu[f].$$
- Theorem 3.28, dominated convergence:
  $f, g, f_1, f_2, \cdots : \Omega \to \overline{\mathbb{R}}$ measurable with $|f_n| \le g$ almost everywhere, $\lim_{n\to\infty} f_n = f$ almost everywhere, and $g \in \mathcal{L}^1(\mu)$. Then,
  $$\lim_{n\to\infty} \mu[f_n] = \mu[f].$$

Monotone Convergence

- Theorem 3.26, monotone convergence:
  $f_1, f_2, \dots \in \mathcal{L}^1(\mu)$ and $f : \Omega \to \overline{\mathbb{R}}$ measurable with $f_n \uparrow f$ almost everywhere. Then,

$$\lim_{n\to\infty} \mu[f_n] = \mu[f].$$

- Proof: $N \in \mathcal{F}$ be such that $\mu(N) = 0$ and $f_n(\omega) \uparrow f(\omega)$ for $\omega \notin N$. Set $g_n := (f_n - f_1)1_{N^c} \geq 0$. This means that $g_n \uparrow (f - f_1)1_{N^c} =: g$ and with Proposition 3.16.2,

$$\mu[f_n] = \mu[f_1] + \mu[g_n] \xrightarrow{n\to\infty} \mu[f_1] + \mu[g] = \mu[f].$$

Lemma von Fatou

- Theorem 3.27: $f_1, f_2, \dots : \Omega \to \overline{\mathbb{R}}_+$ measurable. Then,

$$\liminf_{n\to\infty} \mu[f_n] \geq \mu[\liminf_{n\to\infty} f_n].$$

- Proof: For all $k \geq n$, $f_k \geq \inf_{\ell \geq n} f_\ell$ and thus, for all n,

$$\inf_{k\geq n} \mu[f_k] \geq \mu[\inf_{\ell\geq n} f_\ell].$$

So,

$$\liminf_{n\to\infty} \mu[f_n] = \sup_{n\in\mathbb{N}} \inf_{k\geq n} \mu[f_k] \geq \sup_{n\in\mathbb{N}} \mu[\inf_{k\geq n} f_k] = \mu[\liminf_{n\to\infty} f_n]$$

by monotone convergence.

Dominated convergence

- Theorem 3.28: $f, g, f_1, f_2, \dots : \Omega \to \overline{\mathbb{R}}$ measurable with $|f_n| \le g$ almost everywhere, $\lim_{n\to\infty} f_n = f$ almost everywhere, and $g \in \mathcal{L}^1(\mu)$. Then,

$$\lim_{n\to\infty} \mu[f_n] = \mu[f].$$

- Proof: Wlog, $|f_n| \le g$ and $\lim_{n\to\infty} f_n = f$ everywhere. Use Fatou's lemma and $g - f_n, g + f \ge 0$, i.e.

$$\mu[g+f] \le \liminf_{n\to\infty} \mu[g+f_n] = \mu[g] + \liminf_{n\to\infty} \mu[f_n],$$
$$\mu[g-f] \le \liminf_{n\to\infty} \mu[g-f_n] = \mu[g] - \limsup_{n\to\infty} \mu[f_n].$$

After subtracting $\mu[g]$,

$$\mu[f] \le \liminf_{n\to\infty} \mu[f_n] \le \limsup_{n\to\infty} \mu[f_n] \le \mu[f].$$

Example

- λ: Lebesgue measure, $f_n = 1/n$. Then $f_n \downarrow 0$, but

$$\liminf_{n\to\infty} \mu[f_n] = \infty > 0 = \mu[0] = \mu[\liminf_{n\to\infty} f_n].$$

Example

$|f_n| \leq g \in \mathcal{L}^1(\mu)$ is necessary (here for λ Lebesgue measure)

- $f_n = n \cdot 1_{[0,1/n]} \xrightarrow{n\to\infty} \infty \cdot 1_0$. There is no $g \in \mathcal{L}^1(\lambda)$ with $f_n \leq g$ and

$$\lim_{n\to\infty} \mu[f_n] = 1 \neq 0 = \mu[\lim_{n\to\infty} f_n].$$

- $f_n = n \cdot 1_{[0,1/n^2]} \xrightarrow{n\to\infty} \infty \cdot 1_0$. There is $f_n \leq g \in \mathcal{L}^1(\lambda)$ with

$$\sup_{n\in\mathbb{N}} f_n(x) = \sup\{n : x \leq 1/n^2\} = \left\lfloor \frac{1}{\sqrt{x}} \right\rfloor \leq \frac{1}{\sqrt{x}} =: g(x),$$

and

$$\lim_{n\to\infty} \mu[f_n] = \lim_{n\to\infty} \frac{1}{n} = 0 = \mu[0] = \mu[\lim_{n\to\infty} f_n].$$