Measure Theory for Probabilists
8. Measures on $\mathbb{R}$ and image measures

Peter Pfaffelhuber

January 9, 2024

Lebesgue measure

- Proposition 2.18: There is exactly one measure λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with

$$\lambda((a,b]) = b - a$$

for $a, b \in \mathbb{Q}$ with $a \leq b$.

- Proof: $\mathcal{H} = \{(a,b] : a, b \in \mathbb{Q}, a \leq b\}$ is a semi-ring with $\sigma(\mathcal{H}) = \mathcal{B}(\mathbb{R})$.
σ-additivity: let $a_1, a_2, \dots$ be such that $\bigcup_{n=1}^{\infty}(a_{n+1}, a_n] = (a,b] \in \mathcal{H}$, i.e., $b = a_1$ and $a_n \downarrow a$. Then,

$$\lambda(a,b] = b - a = a_1 - \lim_{N\to\infty} a_N = \sum_{n=1}^{\infty} a_n - a_{n+1} = \sum_{n=1}^{\infty} \lambda((a_{n+1}, a_n]).$$

Conclude with Theorem 2.16.

σ-finite measures on $\mathbb{R}$

- Proposition 2.19: $\mu : \mathcal{B}(\mathbb{R}) \to \overline{\mathbb{R}}_+$ is a σ-finite measure iff there is $G : \mathbb{R} \to \mathbb{R}$, non-decreasing and right-continuous with

$$\mu((a,b]) = G(b) - G(a), \qquad a, b \in \mathbb{Q}, a \leq b. \qquad (*)$$

  If $\widetilde{G}$ also satisfies $(*)$, then $\widetilde{G} = G + c$ for some $c \in \mathbb{R}$.
- Proof: '$\Rightarrow$': Define $G(0) = 0$ and

$$G(x) := \begin{cases} \mu((0,x]), & x > 0, \\ -\mu((x,0]), & x < 0. \end{cases}$$

  '$\Leftarrow$': Similar to the proof of Proposition 2.18.
  Let $\widetilde{G}$ satisfy $(*)$. Then, for $a \in \mathbb{R}$,

$$\widetilde{G}(b) = \widetilde{G}(a) + \mu((a,b]) = G(b) + \widetilde{G}(a) - G(a),$$

  and the assertion follows with $c = \widetilde{G}(a) - G(a)$.

Probability measures on $\mathbb{R}$

- Corollary 2.20: $\mu : \mathcal{B}(\mathbb{R}) \to [0,1]$ is probability measure iff there is $F : \mathbb{R} \to [0,1]$ non-decreasing and right-continuous with $\lim_{b\to\infty} F(b) = 1$ and

  $$\mu((a,b]) = F(b) - F(a), \qquad a,b \in \mathbb{Q}, a \le b.$$

  F is uniquely defined by μ.
  F is called the distribution function of μ.

Examples

- Let $f : \mathbb{R} \to \mathbb{R}_+$ be a density (piecewise continuous with $\int_{-\infty}^{\infty} f(x)dx = 1$). A densitiy defines a distribution function via
$$F(x) := \int_{-\infty}^{x} f(a)da,$$
therefore uniquely a probability measures.

$$F_{U(0,1)}(x) = \int_{-\infty}^{x} 1_{[0,1]}(a)da = \begin{cases} 0, & x \leq 0, \\ x, & 0 < x \leq 1, \\ 1, & x > 1, \end{cases}$$

$$F_{\exp(\lambda)}(x) = \int_{-\infty}^{x} 1_{[0,\infty)}(a)\lambda e^{-\lambda a}da = 1 - e^{-\lambda x}$$

$$F_{N(\mu,\sigma^2)}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} \exp\Big(- \frac{(a-\mu)^2}{2\sigma^2}\Big) da =: \Phi(x)$$

Image measures

- If $\mathcal{F}'$ is a σ-field on Ω', and $f : \Omega \to \Omega'$. Then,
$$\sigma(f) := \{f^{-1}(A') : A' \in \mathcal{F}'\} \text{ is a } \sigma\text{-field on } \Omega.$$
- Definition 2.23: $(\Omega, \mathcal{F}, \mu)$ measure space, $(\Omega', \mathcal{F}')$ measurable space, $f : \Omega \to \Omega'$ with $\sigma(f) \subseteq \mathcal{F}$. Then,
$$\mathcal{F}' \ni A' \mapsto f_*\mu(A') := \mu(f^{-1}(A')) = \mu(f \in A')$$
is the *image measure* of f under μ.
If $\mathbb{P}$ is a probability measure, we call $X_*\mu$ the distribution of X under $\mathbb{P}$.
- Proposition 2.25: $f_*\mu$ is a measure on $\mathcal{F}'$.
- Proof: $A'_1, A'_2, \dots \in \mathcal{F}'$ disjoint, then
$$f_*\mu\Big(\biguplus_{n=1}^{\infty} A'_n\Big) = \mu\Big(f^{-1}\Big(\biguplus_{n=1}^{\infty} A'_n\Big)\Big)$$
$$= \mu\Big(\biguplus_{n=1}^{\infty} (f^{-1}(A'_n)\Big) = \sum_{n=1}^{\infty} \mu(f^{-1}(A'_n)) = \sum_{n=1}^{\infty} f_*\mu(A'_n).$$

Examples

- For $\Omega = [0,1]$, $\mathcal{H} := \{[0,b) : 0 \leq b \leq 1\}$ has $\sigma(\mathcal{H}) = \mathcal{B}([0,1])$.
  $\mu = \mu_{U(0,1)}$, $f : u \mapsto 1 - u$. Then $f_*\mu = \mu$, because

  $$f_*\mu([0,b)) = \mu(f^{-1}([0,b))) = \mu([1-b,1]) = 1 - (1-b) = b.$$

- $\Omega = \mathbb{R}$, $y \in \mathbb{R}$, $f_y : x \mapsto x + y$
  λ Lebesgue measure. Then $(f_y)_*\lambda = \lambda$, because

  $$(f_y)_*\lambda([a,b]) = \lambda(f_y^{-1}([a,b])) = \lambda([a-y, b-y]) = b - a.$$

- $\Omega = [0,1], \Omega' = \mathbb{R}_+$, $f : x \mapsto -\frac{1}{\lambda}\log(x)$ for $\lambda > 0$
  $\mu = \mu_{U(0,1)}$. Then, $f_*\mu = \mu_{\exp(\lambda)}$, because for $x \geq 0$

  $$f_*\mu([0,x]) = \mu(f^{-1}([0,x])) = \mu([e^{-\lambda x}, 1]) = 1 - e^{-\lambda x}.$$