Measure Theory for Probabilists
7. Uniqueness and extension of set functions

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Question

- When does an additive set-function μ on $\mathcal{H}$ uniquely extend to a measure $\widetilde{\mathcal{H}}$ on $\sigma(\mathcal{H})$?
- Uniqueness: Proposition 2.11: Let $\mathcal{C} \subseteq 2^{\Omega}$ be $\cap$-stable, and μ, ν be σ-finite measures on $\sigma(\mathcal{C})$. Then,

$$\mu = \nu \qquad \Longleftrightarrow \qquad \mu|_{\mathcal{C}} = \nu|_{\mathcal{C}}.$$

- Existence: See Carathéodory's Extension Theorem 2.13: Let μ^* be an outer measure. Then, $\mathcal{F}^*$ the set of μ^*-measurable sets is a σ-algebra and $\mu := \mu^*|_{\mathcal{F}^*}$ is a measure.

Theorem 2.16

	Lemma 2.5	Theorem 2.10	Theorem 2.16
μ additive	$\circ$	$\circ$	
μ finite		$\circ$	
μ σ-finite			$\circ$
μ defined on semi-ring	$\circ$	$\circ$	$\circ$
hline μ σ-additive	$\circ/\bullet$	$\bullet$	$\circ$
hline μ σ-subadditive	$\bullet/\circ$		
μ inner $\mathcal{K}$-regular		$\circ$	
μ extends uniquely to $\sigma(\mathcal{H})$			$\bullet$

Proposition 2.11

- Let $\mathcal{C} \subseteq 2^{\Omega}$ be $\cap$-stable, and μ, ν be σ-finite measures on $\sigma(\mathcal{C})$. Then,

$$\mu = \nu \qquad \iff \qquad \mu|_{\mathcal{C}} = \nu|_{\mathcal{C}}.$$

- Proof for finite μ, ν with $\mu(\Omega) = \nu(\Omega)$: $\Rightarrow$: clear
  $\Leftarrow$: Let
  $$\mathcal{D} := \{B \in \mathcal{F} : \mu(A) = \nu(A)\} \supseteq \mathcal{H}.$$
  To show: $\mathcal{D}$ is Dynkin. $\Rightarrow \sigma(\mathcal{H}) \subseteq \mathcal{D}$ by Theorem 1.13.
  - $B, C \in \mathcal{D}, B \subseteq C \Rightarrow \mu(C \setminus B) = \mu(C) - \mu(B) = \nu(C) - \nu(B) = \nu(C \setminus B)$, i.e. $C \setminus B \in \mathcal{D}$.
  - $B_1, B_2, \dots \in \mathcal{D}$ with $B_1 \subseteq B_2 \subseteq B_3 \subseteq \dots \in \mathcal{D}$ and $B = \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$, then from continuity from below,

$$\mu(B) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \nu(B_n) = \nu(B) \quad \Rightarrow B \in \mathcal{D}.$$

Theorem 2.13

- A σ-subadditive, monotone $\mu^* : 2^\Omega \to \mathbb{R}_+$ with $\mu^*(\emptyset) = 0$ is an *outer measure*.
- A set $A \subseteq \Omega$ is called μ^**-measurable* if

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c), \qquad E \subseteq \Omega.$$

- Theorem 2.13: Let μ^* be an outer measure. Then, $\mathcal{F}^*$ the set of μ^*-measurable sets is a σ-algebra and $\mu := \mu^*|_{\mathcal{F}^*}$ is a measure. Furthermore, $\mathcal{N} := \{N \subseteq \Omega : \mu^*(N) = 0\} \subseteq \mathcal{F}^*$.

Theorem 2.13

- Let μ^* be an outer measure. Then, $\mathcal{F}^*$ the set of μ^*-measurable sets is a σ-algebra and $\mu := \mu^*|_{\mathcal{F}^*}$ is a measure.
- Proof: Show:
  - $\emptyset \in \mathcal{F}^*$, since $\mu^*(E) = \mu^*(E \cap \emptyset) + \mu^*(E \cap \Omega)$.
  - $A \in \mathcal{F}^* \Rightarrow A^c \in \mathcal{F}^*$
  - $A, B \in \mathcal{F}^* \Rightarrow A \cap B \in \mathcal{F}^*$, since

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) \cap B^c) + \mu^*(E \cap A^c) \\ &\geq \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \cap B)^c) \geq \mu^*(E),\end{aligned}$$

  - $A_1, A_2, \dots \in \mathcal{F}^*$ disjoint, $B_n = \biguplus_{k=1}^n A_k \in \mathcal{F}^*$, $B_n \uparrow B$. Show $\mu^*(E \cap B_n) = \sum_{k=1}^n \mu^*(E \cap A_k)$ by induction on n:

$$\begin{aligned}\mu^*(E \cap B_{n+1}) &= \mu^*(E \cap B_{n+1} \cap B_n) + \mu^*(E \cap B_{n+1} \cap B_n^c) \\ &= \mu^*(E \cap B_n) + \mu^*(E \cap A_{n+1}) = \sum_{k=1}^{n+1} \mu^*(E \cap A_k).\end{aligned}$$

Theorem 2.13

- Let μ^* be an outer measure. Then, $\mathcal{F}^*$ the set of μ^*-measurable sets is a σ-algebra and $\mu := \mu^*|_{\mathcal{F}^*}$ is a measure.
- Then, $\mu^*(E \cap B) = \sum_{k=1}^{\infty} \mu^*(E \cap A_k) = \lim_{n\to\infty} \mu^*(E \cap B_n)$ since

$$\begin{aligned}\mu^*(E \cap B) &\leq \sum_{k=1}^{\infty} \mu^*(E \cap A_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu^*(E \cap A_k) \\ &= \lim_{n\to\infty} \mu^*(E \cap B_n) \leq \mu^*(E \cap B),\end{aligned}$$

- $B \in \mathcal{F}^*$, since $B_1, B_2, \ldots \in \mathcal{F}^*$, so

$$\begin{aligned}\mu^*(E) &= \lim_{n\to\infty} \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \\ &\geq \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E).\end{aligned}$$

- So, $\mathcal{F}^*$ is a σ-algebra and μ^* is σ-additive on $\mathcal{F}^*$, i.e. $\mu = \mu^*|_{\mathcal{F}^*}$ is a measure.

Theorem 2.13

- $\mathcal{N} := \{N \subseteq \Omega : \mu^*(N) = 0\} \subseteq \mathcal{F}^*$.
- $N \in \mathcal{N}$ are called *(μ^*-)null sets*.
  If $A^c \in \mathcal{N}$, we say that A holds (μ)-*almost everywhere* or *almost surely*.
- Proof: For $N \in \mathcal{N}$, by monotonicity $\mu^*(E \cap N) = 0$, so

$$\begin{aligned}\mu^*(E \cap N^c) + \mu^*(E \cap N) &\geq \mu^*(E) \geq \mu^*(E \cap N^c) \\ &= \mu^*(E \cap N^c) + \mu^*(E \cap N).\end{aligned}$$

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- Test

Zweite Folie

- Test

Proposition 2.8

- μ is σ-additive iff

$$\mu\Big(\biguplus_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

- μ is σ-sub-additive iff

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

- μ is continuous from below, if for $A, A_1, A_2, \ldots$ and $A_1 \subseteq A_2 \subseteq \ldots$ with $A = \bigcup_{n=1}^{\infty} A_n$,

$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

- μ is continuous from above (in the $\emptyset$), if for $A(=\emptyset), A_1, A_2, \ldots,\ \mu(A_1) < \infty$ and $A_1 \supseteq A_2 \supseteq \ldots$ with $A = \bigcap_{n=1}^{\infty} A_n$,

$$(0 =)\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

Proposition 2.8

- Let $\mathcal{R}$ be a ring and $\mu : \mathcal{R} \to \overline{\mathbb{R}}_+$ be additive and $\mu(A) < \infty$ for all $A \in \mathcal{R}$. Then, the following are equivalent:
  1. μ is σ-additive;
  2. μ is σ-subadditive;
  3. μ is continuous from below;
  4. μ is continuous from above in $\emptyset$;
  5. μ is continuous from above.
- Proof: 1.$\Leftrightarrow$2., 5.$\Rightarrow$4.: clear.
  1.$\Rightarrow$3.: With $A_0 = \emptyset$, $A = \biguplus_{n=1}^{\infty} A_n \setminus A_{n-1}$
  3.$\Rightarrow$1.: Set $A_N = \biguplus_{n=1}^{N} B_n$,
  4.$\Rightarrow$5.: With $B_n := A_n \setminus A \downarrow \emptyset$,
  $\mu(A_n) = \mu(B_n) + \mu(A) \xrightarrow{n\to\infty} \mu(A)$.
  3.$\Rightarrow$4.: Set $B_n := A_1 \setminus A_n \uparrow A_1$. Then,
  $\mu(A_1) = \lim_{n\to\infty} \mu(B_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$.
  4.$\Rightarrow$3. Set $B_n := A \setminus A_n \downarrow \emptyset$. Then,
  $0 = \lim_{n\to\infty} \mu(B_n) = \mu(A) - \lim_{n\to\infty} \mu(A_n)$.

Inner regularity of measures on Polish spaces

- Lemma 2.9: $(\Omega, \mathcal{O})$ Polish, μ finite, $\varepsilon > 0$.
  There exists $K \subseteq \Omega$ compact with $\mu(\Omega \setminus K) < \varepsilon$.
- Proof: There is $\{\omega_1, \omega_2, \dots\} \subseteq \Omega$ dense, so $\Omega = \bigcup_{k=1}^{\infty} B_{1/n}(\omega_k)$. μ is continuous from above $\Rightarrow$

$$0 = \mu\Big(\Omega \setminus \bigcup_{k=1}^{\infty} B_{1/n}(\omega_k)\Big) = \lim_{N\to\infty} \mu\Big(\Omega \setminus \bigcup_{k=1}^{N} B_{1/n}(\omega_k)\Big).$$

  Take $N_n \in \mathbb{N}$ with $\mu\Big(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k^n)\Big) < \varepsilon/2^n$ and $A := \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)$ totally bounded, hence relatively compact with

$$\mu(\Omega \setminus \overline{A}) \leq \mu(\Omega \setminus A) \leq \mu\Big(\bigcup_{n=1}^{\infty} \Big(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)\Big)\Big)$$

$$\leq \sum_{n=1}^{\infty} \mu\Big(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)\Big) < \varepsilon.$$

Inner regularity and σ-additivity

- Theorem 2.10: $\mathcal{H}$ semi-ring, $\mu : \mathcal{H} \to \mathbb{R}_+$ finite, finitely additive and inner $\mathcal{K} \subseteq \mathcal{H}$-regular. Then μ is σ-additive.
- Proof:Wlog, $\mathcal{H}$ is ring and $\mathcal{K} = \mathcal{K}_\cup$
  To show: μ is continuous from above in $\emptyset$. Let $A_1, A_2, \dots \in \mathcal{H}$ with $A_1 \supseteq A_2 \supseteq \cdots$ and $\bigcap_{n=1}^\infty A_n = \emptyset$ and $\varepsilon > 0$.
  Choose $K_1, K_2, \dots \in \mathcal{K}$ with $K_n \subseteq A_n, n \in \mathbb{N}$ and

$$\mu(A_n) \leq \mu(K_n) + \varepsilon 2^{-n}.$$

  Then, $\bigcap_{n=1}^\infty K_n \subseteq \bigcap_{n=1}^\infty A_n = \emptyset$, so there is $N \in \mathbb{N}$ with $\bigcap_{n=1}^N K_n = \emptyset$. From this,

$$A_N = A_N \cap \left(\bigcup_{n=1}^N K_n^c \right) = \bigcup_{n=1}^N A_N \setminus K_n \subseteq \bigcup_{n=1}^N A_n \setminus K_n.$$

  By subadditivity and monotonicity of μ, for $m \geq N$,

$$\mu(A_m) \leq \mu(A_N) \leq \sum_{n=1}^N \mu(A_n \setminus K_n) \leq \varepsilon \sum_{n=1}^N 2^{-n} \leq \varepsilon.$$