Measure Theory for Probabilists
4. Dynkin systems and compact systems

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Connections between set-systems

	$\mathcal{C}$ semi-ring	$\mathcal{C}$ ring	$\mathcal{C}$ σ-field
$\mathcal{C}$ is $\cap$-stable	$\bullet$	$\circ$	$\circ$
$\mathcal{C}$ is σ-$\cap$-stable			$\circ$
$\mathcal{C}$ is $\cup$-stable		$\bullet$	$\circ$
$\mathcal{C}$ is σ-$\cup$-stable			$\bullet$
$\mathcal{C}$ is set-difference-stable		$\bullet$	$\circ$
$\mathcal{C}$ is complement-stable			$\bullet$
$B \setminus A = \biguplus_{i=1}^{n} C_i$	$\bullet$	$\circ$	$\circ$
$\Omega \in \mathcal{C}$			$\bullet$

Dynkin systems

- Let $\mathcal{C} \subseteq 2^{\Omega}$. It is often easy to show that $\mathcal{C}$ is a (semi-)ring. However, it is hard to show that $\mathcal{C}$ is a σ-algebra. It is often easier to show that $\mathcal{C}$ is a Dynkin system:
- Definition 1.11: A set system $\mathcal{D}$ is called *Dynkin system* (on Ω) if (i) $\Omega \in \mathcal{D}$, (ii) it is set-difference-stable for subsets (i.e. $A, B \in \mathcal{D}$ and $A \subseteq B$ imply $B \setminus A \in \mathcal{D}$ and (iii) $A_1, A_2, \ldots \in \mathcal{D}$ and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ imply $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$.
- Goal is Theorem 1.13:
  A $\cap$-stable Dynkin system is a σ-algebra.
- Example 1.12:
  $\mathcal{F}$ σ-algebra $\Rightarrow$ $\mathcal{F}$ Dynkin-system
  $\mathcal{F}$ Dynkin system $\Rightarrow$ $\mathcal{F}$ complement-stable

Theorem 1.13:

- $\mathcal{D}$ Dynkin system, $\mathcal{C} \subseteq \mathcal{D}$ is $\cap$-stable $\Rightarrow \sigma(\mathcal{C}) \subseteq \mathcal{D}$.
- Proof: Set

$$\lambda(\mathcal{C}) := \bigcap \{\mathcal{D}' \supseteq \mathcal{C}, \mathcal{D}' \text{ Dynkin-system}\} \supseteq \lambda(\mathcal{C}).$$

Claim: $\lambda(\mathcal{C})$ is a σ-algebra ($\Rightarrow \sigma(\mathcal{C}) \subseteq \sigma(\lambda(\mathcal{C})) = \lambda(\mathcal{C}) \subseteq \mathcal{D}$)
Suffices: $\lambda(\mathcal{C})$ is $\cap$-stable.
Then, $A \cup B = (A^c \cap B^c)^c$, so $\lambda(\mathcal{C})$ is $\cup$-stable and for $A_1, A_2, ... \in \lambda(\mathcal{C})$, we find $\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty \bigcup_{i=1}^n A_i \in \lambda(\mathcal{C})$.
For $B \in \mathcal{C}$, set

$$\mathcal{D}_B := \{A \subseteq \Omega : A \cap B \in \lambda(\mathcal{C})\} \supseteq \mathcal{C}.$$

Then $\mathcal{D}_B$ is a Dynkin system...
So, $\lambda(\mathcal{C}) \subseteq \mathcal{D}_B$. So, for an $A \in \lambda(\mathcal{C})$,

$$\mathcal{B}_A := \{B \subseteq \Omega : A \cap B \in \lambda(\mathcal{C})\} \supseteq \lambda(\mathcal{C}) \text{ is Dynkin system.}$$

Compact sets

- $J \subseteq_f I$ if $J \subseteq I$ and J is finite
- Definition A.7: (Ω, r) metric space, $K \subseteq \Omega$.
  1. K is *compact* if every open cover has a finite partial cover: If $O_i \in \mathcal{O}, i \in I$ and $K \subseteq \bigcup_{i \in I} O_i$, then there is $J \subset_f I$ with $K \subseteq \bigcup_{i \in J} O_i$.
  2. K is *relatively compact* if $\overline{K}$ is compact.
  3. K is *relatively sequentially compact* if for every sequence in K there is a convergent subsequence.
  4. $K \subseteq \Omega$ is *totally bounded* if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ and $\omega_1, \ldots, \omega_N \in K$ such that $K \subseteq \bigcup_{n=1}^{N} B_\varepsilon(\omega_n)$.
- Lemma A.8:: $K \subseteq \Omega$ compact $\Rightarrow$ K is closed.

Compact sets

- Proposition A.9: $K \subseteq \Omega$.
  1. K is relatively compact.
  2. If $F_i \subseteq \overline{K}$ is closed, $i \in I$, and $\bigcap_{i \in I} F_i = \emptyset$, then there is $J \subseteq_f I$ with $\bigcap_{i \in J} F_i = \emptyset$.
  3. K is relatively sequentially compact.
  4. K is totally bounded.

  Then
  $$4. \Longleftarrow 1. \iff 2. \Longrightarrow 3.$$
  Furthermore, 3. $\Longrightarrow$ 2. also holds if $(\Omega, \mathcal{O})$ is separable and 4. $\Longrightarrow$ 3. if (Ω, r) is complete.

Compact systems

- Definition 1.14: $\mathcal{K}$ $\cap$-stable is *compact system* if $\bigcap_{n=1}^{\infty} K_n = \emptyset$ with $K_1, K_2, \ldots \in \mathcal{K}$ implies that there is a $N \in \mathbb{N}$ with $\bigcap_{n=1}^{N} K_n = \emptyset$.
- Example 1.15: $\mathcal{K} \subseteq \{K \subseteq \Omega : K \text{ compact}\}$ $\cap$-stable is compact system.
  Indeed: Let $\bigcap_{n=1}^{\infty} K_n = \emptyset$. Then, K_1 and $L_n := K_1 \cap K_n \subseteq K_1$ are compact and (because of the compactness of K_1) there is an N with $\bigcap_{n=1}^{N} K_n = \emptyset$ due to Proposition A.9.

Compact systems

- Lemma 1.16: $\mathcal{K}$ compact system. Then,

$$\mathcal{K}_\cup := \left\{ \bigcup_{i=1}^n K_i : K_1, \dots, K_n \in \mathcal{K}, n \in \mathbb{N} \right\}$$

  is also a compact system.

- Proof: $\mathcal{K}_\cup$ is $\cap$-stable. Let $L_1 = \bigcup_{j=1}^{m_1} K_j^1, L_2 = \bigcup_{j=1}^{m_2} K_j^2, \dots \in \mathcal{K}_\cup$ with $\bigcap_{n=1}^N L_n \neq \emptyset$ for all $N \in \mathbb{N}$. To show: $\bigcap_{n=1}^\infty L_n \neq \emptyset$. Use induction over N for:

  For every $N \in \mathbb{N}$ there are sets $K_1, \dots, K_N \in \mathcal{K}$ with $K_n \subseteq L_n$, $n = 1, \dots, N$, such that for all $k \in \mathbb{N}_0$ we have $K_1 \cap \dots \cap K_N \cap L_{N+1} \cap \dots \cap L_{N+k} \neq \emptyset$.

  Then, use $k = 0$. So we see that there are $K_1, K_2, \dots \in \mathcal{K}$ and $K_n \subseteq L_n$, $n \in \mathbb{N}$ with $\bigcap_{n=1}^N K_n \neq \emptyset$ for all $N \in \mathbb{N}$. Hence, $\emptyset \neq \bigcap_{n=1}^\infty K_n \subseteq \bigcap_{n=1}^\infty L_n$.