# Measure Theory for Probabilists
# 3. Generators and extensions

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# Generated ring/$\sigma$-algebra

- Let $\mathcal{C} \subseteq 2^{\Omega}$. Then,

$$\mathcal{R}(\mathcal{C}) := \bigcap \Big\{ \mathcal{R} \supseteq \mathcal{C} : \mathcal{R} \text{ ring} \Big\},$$
$$\sigma(\mathcal{C}) := \bigcap \Big\{ F \supseteq \mathcal{C} : \mathcal{F} \ \sigma\text{-field} \Big\}$$

  are the ring and $\sigma$-algebra generated from $\mathcal{C}$,
- Example 1.6: Let $\mathcal{H} := \{[a, b), a \leq b, a, b \in \mathbb{Q}\}$. Then,

$$\mathcal{R}(\mathcal{H}) = \Big\{ \biguplus_{k=1}^{n} (a_k, b_k] : a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{Q}, \\ a_k < b_k, k = 1, \dots, n \text{ and } a_k < b_{k+1}, k = 1, \dots, n-1 \Big\}$$

  is the ring generated from $\mathcal{H}$.

# Generated ring

- Lemma 1.5: $\mathcal{H}$ semi-ring. Then,

$$\mathcal{R}(\mathcal{H}) = \left\{ \biguplus_{k=1}^{n} A_k : A_1, \ldots, A_n \in \mathcal{H} \text{ disjoint}, n \in \mathbb{N} \right\}$$

  is the ring generated from $\mathcal{H}$.

- Proof: $\mathcal{R}(\mathcal{H})$ is $\cap$-stable.
  To show: $\mathcal{R}(\mathcal{H})$ set-difference-stable. Let $A_1, \ldots, A_n \in \mathcal{H}$ and $B_1, \ldots, B_m \in \mathcal{H}$ be disjoint. Then,

$$\left( \biguplus_{i=1}^{n} A_i \right) \setminus \left( \biguplus_{j=1}^{m} B_j \right) = \biguplus_{i=1}^{n} \bigcap_{j=1}^{m} A_i \setminus B_j \in \mathcal{R}(\mathcal{H}).$$

  To show: $\mathcal{R}(\mathcal{H})$ is $\cup$-stable:

$$A \cup B = (A \cap B) \uplus (A \setminus B) \uplus (B \setminus A) \in \mathcal{R}(\mathcal{H})$$

# Definitions from topology

- $\Omega$ some set. A set system $\mathcal{O} \subseteq 2^\Omega$ is called *topology* if (i) $\emptyset, \Omega \in \mathcal{O}$; (ii) if $\mathcal{O}$ is $\cap$-stable; (iii) if $I$ is arbitrary and if $A_i \in \mathcal{O}, i \in I$, then $\bigcup_{i \in I} A_i \in \mathcal{O}$. The pair $(\Omega, \mathcal{O})$ is called *topological space*. Its members, i.e. every $A \in \mathcal{O}$, is called *open*; any set $A \subseteq \Omega$ with $A^c \in \mathcal{O}$ is called *closed*.
- $(\Omega, r)$ be a metric space and $B_\varepsilon(\omega) := \{\omega' \in \Omega : r(\omega, \omega') < \varepsilon\}$ an open ball and

$$\mathcal{B} := \{B_\varepsilon(\omega) : \varepsilon > 0, \omega \in \Omega\}. \tag{1}$$

Then,

$$\begin{aligned} \mathcal{O}(\mathcal{B}) &:= \{A \subseteq \Omega : \quad \forall \omega \in A \, \exists B \in \mathcal{B} : \omega \in B \subseteq A\} \\ &= \Big\{ \bigcup_{B \in \mathcal{C}} B : \mathcal{C} \subseteq \mathcal{B} \Big\} \end{aligned}$$

is the topology generated by $r$.

# Definitions from topology

- $r$ is called *complete*, if every Cauchy-sequence converges.
- If there is some countable $\Omega'$ such that $\inf_{x' \in \Omega'} r(x, x') = 0$ for all $x \in \Omega$, we call $(\Omega, r)$ separable. In this case,
$$\mathcal{B}' := \{B_r(\omega') : \omega' \in \Omega', r \in \mathbb{Q}_+\}$$
is countable and $\mathcal{O}(\mathcal{B}') = \mathcal{O}(\mathcal{B})$.
- The space $(\Omega, \mathcal{O})$ is called Polish, if it is separable and completely metrizable.

# Borel's $\sigma$-field

- Definition 1.7: $(\Omega, \mathcal{O})$ a topological space.

$$\mathcal{B}(\Omega) := \sigma(\mathcal{O})$$

  is the *Borel $\sigma$-algebra* on $\Omega$. Sets in $\mathcal{B}(\Omega)$ are also called *(Borel-)measurable sets*.
- Lemma 1.8: Let $(\Omega, \mathcal{O})$ be a topological space with countable basis $\mathcal{C} \subseteq \mathcal{O}$. Then, $\sigma(\mathcal{O}) = \sigma(\mathcal{C})$.
- Proof: To show $\mathcal{O} \subseteq \sigma(\mathcal{C})$. Clear, since any $A \in \mathcal{O}$ can be represented as a countable union of sets from $\mathcal{C}$.

# Borel $\sigma$-field generated by interavls

- Lemma 1.9: The set system

$$\mathcal{C}_1 = \{[-\infty, b] : b \in \mathbb{Q}\}$$

  generates $\mathcal{B}(\mathbb{R})$.
- Proof: Generate $(a, b]$ from $[-\infty, b] \setminus [-\infty, a]$, then $(a, b) = \bigcup_{i=1}^{\infty}(a, b - \frac{1}{n})$. These sets clearly generate $\mathcal{B}(\mathbb{R})$.