Basics in Applied Mathematics

Part I: Stochastics — Winter semester 2026/2027

Exercise Sheet 4

Continuous random variables: densities, expectation and variance, the central limit theorem; the gamma distribution

Hand in: ____________________ Each exercise is worth 4 points.

Exercise 1 **(4 points)**

Densities, distribution functions, expectation and variance.

(a) Let $X \sim U([a,b])$ have density $f(x) = \frac{1}{b-a} 1_{[a,b]}(x)$. Determine the distribution function F_X, verify $\int_{\mathbb{R}} f = 1$, and compute $\mathbf{E}[X]$ and $\mathbf{Var}[X]$.

(b) Let $X \sim \exp(\lambda)$ have distribution function $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. Find the density, and compute $\mathbf{E}[X]$ and $\mathbf{Var}[X]$ using the tail formulas $\mathbf{E}[X] = \int_0^\infty \mathbf{P}(X > x)\, dx$ and $\mathbf{E}[X^2] = 2\int_0^\infty x\, \mathbf{P}(X > x)\, dx$.

(c) Let $f(x) = c\,x^2$ for $x \in [0,1]$ and $f(x) = 0$ otherwise. Determine the constant c, the distribution function, the median, and $\mathbf{E}[X]$.

(d) Show that the exponential distribution is memoryless: for all $s,t \geq 0$, $\mathbf{P}(X > s+t \mid X > t) = \mathbf{P}(X > s)$.

Exercise 2 **(4 points)**

Transformations and inverse-transform sampling.

(a) Let $U \sim U([0,1])$. Show that $-\frac{1}{\lambda} \ln U \sim \exp(\lambda)$.

(b) (Inverse transform.) Let F be a continuous, strictly increasing distribution function and $U \sim U([0,1])$. Show that $F^{-1}(U)$ has distribution function F. (This is how one simulates a general continuous distribution from a uniform random number.)

(c) Let $X \sim \exp(\lambda)$. Show that $\lfloor X \rfloor$ (the integer part) is geometrically distributed, and identify its parameter.

(d) Let $X \sim U([0,1])$. Compute the density of $Y = X^2$.

Exercise 3 **(4 points)**

The normal distribution and the central limit theorem.

(a) Let $Z \sim N(0,1)$. Assuming $\int_{-\infty}^\infty e^{-x^2/2}\, dx = \sqrt{2\pi}$, show by symmetry and integration by parts that $\mathbf{E}[Z] = 0$ and $\mathbf{Var}[Z] = 1$.

(b) Let $X = \mu + \sigma Z$ with $Z \sim N(0,1)$ and $\sigma > 0$. Show that $X \sim N(\mu, \sigma^2)$ and compute $\mathbf{E}[X]$, $\mathbf{Var}[X]$.

(c) A fair die is rolled 100 times; let S be the sum of the pips. Using $\mathbf{E}[S] = 350$ and $\mathbf{Var}[S] = \frac{3500}{12}$, approximate $\mathbf{P}(S \geq 380)$ with the central limit theorem.

(d) (de Moivre–Laplace.) Approximate $\mathbf{P}(\mathrm{Bin}(100, \frac{1}{2}) \geq 60)$ by a normal distribution, once without and once with a continuity correction, and compare the two.

Exercise 4 **(4 points)**

Sums, minima and the gamma distribution. Let $X_1, X_2, \ldots$ be i.i.d. $\exp(\lambda)$.

(a) Show, by computing the convolution, that $X_1 + X_2$ has density $f(x) = \lambda^2 x\, e^{-\lambda x}$ for $x \geq 0$; this is the gamma density $\Gamma(2, \lambda)$.

(b) By induction, show that $X_1 + \cdots + X_\alpha \sim \Gamma(\alpha, \lambda)$ with density $\frac{\lambda^\alpha}{(\alpha-1)!} x^{\alpha-1} e^{-\lambda x}$, and deduce $\mathbf{E}[X_1 + \cdots + X_\alpha] = \alpha/\lambda$ and variance α/λ^2.

(c) Show that $\min(X_1, \ldots, X_n) \sim \exp(n\lambda)$, and hence $\mathbf{E}[\min(X_1, \ldots, X_n)] = \frac{1}{n\lambda}$.

(d) Let X, Y be i.i.d. $\exp(1)$ and set $U = X + Y$, $V = \frac{X}{X+Y}$. Determine the joint density of (U, V), and deduce that U and V are independent with $V \sim U([0,1])$.