Basics in Applied Mathematics

Part I: Stochastics — Winter semester 2026/2027

Exercise Sheet 3

Expectation, variance and covariance; geometric, negative binomial, Poisson and multinomial distributions

Hand in: ____________________ Each exercise is worth 4 points.

Throughout, $q := 1 - p$.

Exercise 1 **(4 points)**

Waiting times: geometric and negative binomial. Let $X \sim \mathrm{geo}(p)$ be the number of independent $\mathrm{Ber}(p)$ trials up to and including the first success, so $\mathbf{P}(X = k) = q^{k-1}p$ for $k = 1, 2, \ldots$

(a) Using the tail formula $\mathbf{E}[X] = \sum_{x \geq 0} \mathbf{P}(X > x)$ together with $\mathbf{P}(X > x) = q^x$, compute $\mathbf{E}[X]$.

(b) Using $\mathbf{E}[X(X-1)] = 2 \sum_{x \geq 0} x \, \mathbf{P}(X > x)$, compute $\mathbf{Var}[X]$.

(c) Show that X is *memoryless*: for all $k, \ell \in \mathbb{N}$, $\mathbf{P}(X > k + \ell \mid X > \ell) = \mathbf{P}(X > k)$.

(d) Let $X \sim \mathrm{NB}(r, p)$ be the number of trials up to the r-th success. Writing $X = Y_1 + \cdots + Y_r$ with independent $Y_i \sim \mathrm{geo}(p)$, deduce $\mathbf{E}[X] = r/p$ and $\mathbf{Var}[X] = rq/p^2$. Evaluate both for the number of rolls of a fair die until the third six.

Exercise 2 **(4 points)**

The Poisson distribution.

(a) For $X \sim \mathrm{Poi}(\lambda)$, compute $\mathbf{E}[X]$ and $\mathbf{Var}[X]$ (both equal to λ).

(b) Let $X \sim \mathrm{Poi}(\lambda)$ and $Y \sim \mathrm{Poi}(\mu)$ be independent. Show that $X + Y \sim \mathrm{Poi}(\lambda + \mu)$.

(c) Prove the Poisson limit theorem: if $X_n \sim \mathrm{Bin}(n, p_n)$ with $n\, p_n \to \lambda > 0$, then $\mathbf{P}(X_n = k) \to e^{-\lambda}\lambda^k/k!$ for every k.

(d) (Balls in bins.) Throw n balls independently and uniformly into n bins, and let Z be the number of empty bins. Using indicator variables and linearity of the expectation, show $\mathbf{E}[Z] = n\left(1 - \frac{1}{n}\right)^n$, and deduce $\mathbf{E}[Z]/n \to e^{-1}$ as $n \to \infty$.

Exercise 3 **(4 points)**

Covariance, the multinomial distribution and Chebyshev. Let $(X_1, \ldots, X_\ell) \sim \mathrm{Mult}(n; p_1, \ldots, p_\ell)$ record how n independent trials fall into ℓ categories (category i with probability p_i, $\sum_i p_i = 1$).

(a) Show that each coordinate satisfies $X_i \sim \mathrm{Bin}(n, p_i)$, and hence $\mathbf{E}[X_i] = np_i$, $\mathbf{Var}[X_i] = np_i(1 - p_i)$.

(b) For $i \neq j$, show $\mathbf{Cov}[X_i, X_j] = -np_ip_j$ (write $X_i = \sum_{t=1}^n A_t^{(i)}$ with $A_t^{(i)}$ the indicator that trial t falls in category i). Explain the sign.

(c) Compute the correlation coefficient $\mathbf{Cor}[X_i, X_j]$.

(d) For $S_n \sim \mathrm{Bin}(n, p)$, use the Chebyshev inequality to bound $\mathbf{P}\left(|\frac{S_n}{n} - p| \geq \varepsilon\right)$, and conclude the weak law of large numbers $\mathbf{P}\left(|\frac{S_n}{n} - p| \geq \varepsilon\right) \to 0$ for every $\varepsilon > 0$.

Exercise 4 **(4 points)**

Random sums and conditional expectation. Let N be a $\mathbb{Z}_+$-valued random variable with finite mean and variance, and let $X_1, X_2, \ldots$ be i.i.d. with mean $\mu := \mathbf{E}[X_1]$ and variance $\sigma^2 := \mathbf{Var}[X_1]$, independent of N. Put $S = \sum_{i=1}^N X_i$ (with $S = 0$ if $N = 0$).

(a) Using the tower property, show *Wald's identity* $\mathbf{E}[S] = \mathbf{E}[N]\,\mu$.

(b) Show $\mathbf{E}[S^2 \mid N] = N\sigma^2 + N^2\mu^2$, and deduce the *law of total variance* $\mathbf{Var}[S] = \mathbf{E}[N]\,\sigma^2 + \mu^2\,\mathbf{Var}[N]$.

(c) Specialise to $N \sim \mathrm{Poi}(\lambda)$ and show $\mathbf{E}[S] = \lambda\mu$ and $\mathbf{Var}[S] = \lambda\,\mathbf{E}[X_1^2]$.

(d) A shop is visited by a $\mathrm{Poi}(\lambda)$ number of customers, each spending an independent amount with mean μ and variance σ^2. Give the mean and standard deviation of the total revenue, and evaluate them for $\lambda = 20$, $\mu = 15$, $\sigma = 5$.