Basics in Applied Mathematics

Part I: Stochastics — Winter semester 2026/2027

Exercise Sheet 2

Independence, product spaces, conditional probability; Bernoulli, binomial and hypergeometric distributions

Hand in: ____________________ Each exercise is worth 4 points.

Exercise 1 **(4 points)**

Independence of events and product spaces. A fair coin and a fair die are thrown, independently of each other. We model this by the product space $\Omega = \{H, T\} \times \{1, \dots, 6\}$.

(a) Give the probability distribution $\mathbf{P}$ on Ω explicitly, and compute the probability of the event "heads *and* the die shows at least a 5".

(b) Let $A = \{\text{heads}\}$ and $B = \{\text{the die shows a prime number}\}$. Show that A and B are independent.

(c) Prove in general: if two events A, B are independent, then so are A^c and B, and so are A^c and B^c.

(d) Toss a fair coin twice and let $A = \{\text{first toss heads}\}$, $B = \{\text{second toss heads}\}$, $C = \{\text{the two tosses differ}\}$. Show that A, B, C are *pairwise* independent but not (jointly) independent.

Exercise 2 **(4 points)**

Conditional probability and Bayes' formula. An urn contains three coins: two fair coins and one biased coin that shows heads with probability $p \in (0, 1)$, $p \neq \frac{1}{2}$. A coin is drawn uniformly at random and then tossed twice; call the outcomes X_1, X_2.

(a) Using the law of total probability, compute $\mathbf{P}(X_1 = X_2 = \text{heads})$.

(b) Given that both tosses show heads, use Bayes' formula to compute the probability that the biased coin was drawn.

(c) Show that X_1 and X_2 are *not* independent by verifying $\mathbf{P}(X_2 = \text{heads} \mid X_1 = \text{heads}) \neq \mathbf{P}(X_2 = \text{heads})$. Explain intuitively why observing X_1 changes the distribution of X_2, even though, *given the coin type*, the two tosses are independent.

Exercise 3 **(4 points)**

Three distributions and their normalisation.

(a) (Binomial.) Show that $\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = 1$ (with $q = 1 - p$), so that $\mathrm{Bin}(n, p)$ is indeed a probability distribution.

(b) (Hypergeometric.) An urn holds g balls, w of them white. Draw n balls without replacement and let X be the number of white balls drawn. Show that $\mathbf{P}(X = k) = \binom{w}{k}\binom{g-w}{n-k} / \binom{g}{n}$ and that these probabilities sum to 1. *Hint: Vandermonde's identity* $\sum_k \binom{w}{k}\binom{g-w}{n-k} = \binom{g}{n}$.

(c) Explain why drawing the n balls *with* replacement instead gives $X \sim \mathrm{Bin}(n, w/g)$.

(d) Prove the limit relation between the two models: if $g, w \to \infty$ with $w/g \to p \in (0, 1)$ and n fixed, then $\mathbf{P}(\mathrm{Hyp}(n, g, w) = k) \to \mathbf{P}(\mathrm{Bin}(n, p) = k)$ for every k.

Exercise 4 **(4 points)**

Sums and conditioning for binomials. Let $X \sim \mathrm{Bin}(n, p)$ and $Y \sim \mathrm{Bin}(m, p)$ be *independent*, with the same success probability p, and write $q = 1 - p$.

(a) Prove that $X+Y \sim \mathrm{Bin}(n+m, p)$. *Hint: condition on the value of X (law of total probability), then use Vandermonde's identity.*

(b) Show that, conditionally on $X + Y = k$, the variable X is hypergeometrically distributed:

$$\mathbf{P}(X = \ell \mid X + Y = k) = \frac{\binom{n}{\ell}\binom{m}{k-\ell}}{\binom{n+m}{k}}, \qquad \ell = 0, \ldots, k.$$

Note that the parameter p has disappeared.

(c) Let $Z_1, \ldots, Z_n$ be independent $\mathrm{Ber}(p)$ variables and $S = Z_1 + \cdots + Z_n$. Show that, given $S = k$, the vector $(Z_1, \ldots, Z_n)$ is uniformly distributed over the $\binom{n}{k}$ zero-one vectors with exactly k ones; deduce $\mathbf{P}(Z_1 = 1 \mid S = k) = k/n$ and interpret this result.