Basics in Applied Mathematics

Part I: Stochastics — Winter semester 2026/2027

Exercise Sheet 1

Combinatorics and the configuration model for random graphs

Hand in: ____________________ Each exercise is worth 4 points.

Recall the Erdős–Rényi random graph $G(n,p)$ from the lecture, in which every possible edge is present independently with probability p. In this sheet we study a different, very flexible model of a random graph that lets us *prescribe* the degrees.

The configuration model. Fix $n \geq 1$ and a *degree sequence* $\underline{d} = (d_1, \ldots, d_n)$ of non-negative integers with even sum $\sum_{i=1}^n d_i = 2m$. Attach to each vertex $i \in \{1, \ldots, n\}$ exactly d_i *half-edges* (or *stubs*); there are $2m$ stubs in total. A *configuration* is a perfect matching of these $2m$ stubs into m unordered pairs. Choosing a configuration *uniformly at random* and drawing an edge for every matched pair of stubs yields a random (multi-)graph $\mathrm{CM}_n(\underline{d})$, the *configuration model.* The resulting graph may have *self-loops* (two stubs of the same vertex matched together) and *multiple edges.*

Exercise 1 **(4 points)**

Counting configurations.

(a) Show that the number of perfect matchings of $2m$ labelled stubs is

$$(2m-1)!! := (2m-1)(2m-3)\cdots 3 \cdot 1 = \frac{(2m)!}{2^m\, m!}.$$

Hint: match the stubs one pair at a time; or count ordered sequences of the $2m$ *stubs and divide out the over-counting.*

(b) Evaluate $(2m-1)!!$ for $m = 1, 2, 3$.

(c) Fix one particular stub s. Show that, under the uniform distribution on configurations, s is matched to any prescribed other stub s' with probability $\frac{1}{2m-1}$.

(d) For two *disjoint* pairs of stubs $\{s_1, s_1'\}$ and $\{s_2, s_2'\}$, compute the probability that s_1 is matched to s_1' *and* s_2 is matched to s_2'.

Exercise 2 **(4 points)**

Self-loops. For a vertex i let L_i be the event that $\mathrm{CM}_n(\underline{d})$ has at least one self-loop at i.

(a) For two distinct stubs a, b at vertex i, show $\mathbf{P}(a \text{ is matched to } b) = \frac{1}{2m-1}$. How many (unordered) pairs of stubs are there at vertex i?

(b) Using the inclusion–exclusion formula (or a union bound), show

$$\mathbf{P}(L_i) \;\leq\; \binom{d_i}{2} \frac{1}{2m-1}.$$

(c) Consider the d-regular case $d_1 = \cdots = d_n = d$ (so $2m = nd$). Show that the probability that $\mathrm{CM}_n(\underline{d})$ has *any* self-loop satisfies

$$\mathbf{P}\Big(\bigcup_{i=1}^n L_i\Big) \;\leq\; \frac{n\binom{d}{2}}{nd-1} \xrightarrow{n\to\infty} \frac{d-1}{2}.$$

In particular the number of self-loops does not blow up as $n \to \infty$, but stays of order 1.

(d) For a vertex i of degree $d_i = 2$, compute the *exact* probability that its two stubs form a self-loop.

Exercise 3 **(4 points)**

Edges between two vertices, and a comparison with $G(n,p)$. Fix two distinct vertices $i \neq j$.

(a) Show that a given stub of i is matched to a given stub of j with probability $\frac{1}{2m-1}$, and that there are $d_i d_j$ such ordered pairs (stub of i, stub of j).

(b) Let E_{ij} be the event that i and j are joined by at least one edge. Using a union bound, show $\mathbf{P}(E_{ij}) \leq \dfrac{d_i d_j}{2m-1}$, and argue that $\mathbf{P}(E_{ij}) \approx \dfrac{d_i d_j}{2m-1}$ when the degrees are small compared with m.

(c) Show that the probability of a *double* edge between i and j (two stubs of i matched to two stubs of j) is of order $O(1/m^2)$; more precisely, bound it by $\binom{d_i}{2}\binom{d_j}{2}\dfrac{2}{(2m-1)(2m-3)}$.

(d) In the d-regular case ($2m = nd$), deduce from (b) that $\mathbf{P}(E_{ij}) \approx \frac{d}{n}$. Compare this with the Erdős–Rényi graph $G(n,p)$ with $p = d/n$ (same expected degree), and comment on the similarities and differences of the two models.

Exercise 4 **(4 points)**

Uniformity over simple graphs. Call a graph *simple* if it has neither self-loops nor multiple edges. Let H be a fixed *simple* graph on the vertex set $\{1,\dots,n\}$ whose degree sequence equals $\underline{d}$.

(a) Show that the number of configurations (stub-matchings) that produce exactly the graph H equals $\prod_{i=1}^n d_i!$. *Hint: independently permute the d_i stubs at each vertex i; check that each of these $\prod_i d_i!$ stub-labellings yields a distinct configuration, and that every configuration giving H arises in this way.*

(b) Deduce that
$$\mathbf{P}\big(\mathrm{CM}_n(\underline{d}) = H\big) = \frac{\prod_{i=1}^n d_i!}{(2m-1)!!},$$
and that this value is the *same* for every simple graph H with degree sequence $\underline{d}$. Conclude that, conditioned on the event $\{\mathrm{CM}_n(\underline{d}) \text{ is simple}\}$, the configuration model is *uniformly distributed* over all simple graphs with degree sequence $\underline{d}$.

(c) Let $S(\underline{d})$ be the number of simple graphs on $\{1,\dots,n\}$ with degree sequence $\underline{d}$. Express $\mathbf{P}\big(\mathrm{CM}_n(\underline{d}) \text{ is simple}\big)$ in terms of $S(\underline{d})$, $\prod_i d_i!$ and $(2m-1)!!$.

(d) Let $n = 4$ and $\underline{d} = (2,2,2,2)$. List all simple graphs on $\{1,2,3,4\}$ with this degree sequence, determine $S(\underline{d})$, and compute the probability that $\mathrm{CM}_4(2,2,2,2)$ is simple.