Exercise Sheet 5

Numerical integration and ordinary differential equations (Chapters 7 and 8)

Discussed in the tutorials of Friday 27 November 2026

Exercise 1 (Symmetric rules and Gaussian quadrature) — 3 points

For $a \in (0, 1]$ and weights $w, w_0 \in \mathbb{R}$ consider the quadrature rule

$$Q_a(f) = w\, f(-a) + w_0\, f(0) + w\, f(a) \qquad \text{for} \qquad I(f) = \int_{-1}^{1} f(x)\,\mathrm{d}x.$$

(a) [1 P] Explain why Q_a integrates the odd monomials $x, x^3, x^5, \ldots$ exactly, whatever w and w_0 are. Show that Q_a is exact of degree 3, that is, exact for all polynomials of degree at most 3 (Definition 7.1), if and only if $w = \frac{1}{3a^2}$ and $w_0 = 2 - \frac{2}{3a^2}$.

(b) [1 P] For which a is the rule of (a) exact of degree 5? Compute its weights and show that it is not exact of degree 6. (Its nodes are the roots of the Legendre polynomial $P_3(x) = \frac{1}{2}(5x^3 - 3x)$: this is the Gauss–Legendre rule of Theorem 7.8 with $n = 2$.)

(c) [1 P] Show that for $a = \frac{1}{\sqrt{3}}$ the rule of (a) is $Q(f) = f\big(-\frac{1}{\sqrt{3}}\big) + f\big(\frac{1}{\sqrt{3}}\big)$, the two-point Gauss–Legendre rule (its nodes are the roots of $P_2(x) = \frac{1}{2}(3x^2 - 1)$). Show that it is exact of degree 3 but not of degree 4.

Exercise 2 (The composite trapezoidal rule and the EOC) — 3 points

Let $I := \int_0^1 \frac{\mathrm{d}x}{1+x} = \ln 2 \approx 0.693147$, and let T_h denote the composite trapezoidal rule (Section 7.2) with N subintervals of length $h = \frac{1}{N}$.

(a) [1 P] Compute T_h for $N = 1, 2, 4$ as fractions, the errors $e_h := |I - T_h|$, and the experimental orders of convergence EOC (Section 7.2) for the pairs of step sizes $(1, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{4})$.

(b) [1 P] Which signs of $I - T_h$ and of $I - Q_{\mathrm{M}}$, where $Q_{\mathrm{M}} = f(\frac{1}{2})$ is the midpoint rule for $f(x) = \frac{1}{1+x}$ on $[0, 1]$, do Theorem 7.5 and Theorem 7.3(i) predict, and why? How large must N be for the bound of Theorem 7.5 to guarantee $|I - T_h| \le 10^{-4}$?

(c) [1 P] Now let $I := \int_0^1 \sqrt{x}\,\mathrm{d}x = \frac{2}{3}$. For $N = 1, 2, 4, 8$ the errors $I - T_h$ are approximately 0.1667, 0.0631, 0.0234 and 0.00854. Compute the three EOCs. Show with the substitution $x = hs$ that the error of the trapezoidal rule on the first subinterval $[0, h]$ equals $\frac{1}{6}h^{3/2}$. The errors on the other subintervals are positive, since $\sqrt{x}$ is concave (Theorem 7.3(ii)). What does this imply for the order of convergence, and why does Theorem 7.5 not apply?

Exercise 3 (Lipschitz condition, uniqueness, Picard iteration) — 3 points

(a) [1 P] Which of the following functions $f\colon [0, 1] \times \mathbb{R} \to \mathbb{R}$ are globally Lipschitz continuous in y as in Theorem 8.3? Give a Lipschitz constant L or show that none exists.

$$f_1(t, y) = t^2 \sin y + e^t, \qquad f_2(t, y) = |y| + t, \qquad f_3(t, y) = y^3.$$

(b) [1 P] Show that the initial value problem $y' = 3|y|^{2/3}$, $y(0) = 0$, has at least two solutions $y \in C^1([0, 1])$. (Hint: try $y(t) = t^p$.) Which assumption of Theorem 8.3 is violated?

(c) [1 P] For $y' = 2ty$, $y(0) = 1$, on $[0, 1]$, compute the Picard iterates $y^{(1)}, y^{(2)}, y^{(3)}$ of the proof of Theorem 8.3, $y^{(k+1)}(t) = 1 + \int_0^t 2s\, y^{(k)}(s)\, \mathrm{d}s$, starting with $y^{(0)}(t) = 1$. Guess the limit and verify that it solves the initial value problem. Give a Lipschitz constant of $f(t, y) = 2ty$ on $[0, 1] \times \mathbb{R}$.

Exercise 4 (Euler's and Heun's methods) 3 points

(a) [1 P] Consider $y' = t - y$, $y(0) = 1$, with the exact solution $y(t) = t - 1 + 2e^{-t}$. Carry out two steps of the explicit Euler method (Algorithm 8.5) and two steps of Heun's method (Section 8.4) with $h = \frac{1}{2}$, as fractions, and compare the errors $|y_2 - y(1)|$, where $y(1) = \frac{2}{e} \approx 0.7358$. How many evaluations of f does each method use?

(b) [1 P] Now consider $y' = \lambda y$, $y(t_0) = y_0$, on $[t_0, T]$ with $\lambda \in \mathbb{R}$, the step size $h = (T - t_0)/N$ and the grid points $t_n = t_0 + nh$, $n = 0, \ldots, N$. Show that one step of Heun's method is

$$y_{n+1} = \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right) y_n,$$

and determine the increment function Φ (Definition 8.6).

(c) [1 P] For the initial value problem of (b), compute the local truncation error $\tau_n(h)$ of Definition 8.6, using $y(t_{n+1}) = e^{\lambda h} y(t_n)$ and $e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + O(z^4)$. Conclude that Heun's method is consistent of order 2 for this initial value problem. What is the order in h of the error of a single step started from the exact value $y(t_n)$?