Exercise Sheet 4

Eigenvalue problems and polynomial interpolation (Chapters 5 and 6)

Handed out Thursday 12 November 2026 · discussed in the tutorials of Friday 20 November 2026

15 points, about two and a half hours

The sheet is not handed in. All exercises are to be solved by hand, in exact arithmetic (fractions and square roots are left as they are). Points indicate the weight of a part. Result numbers refer to the lecture notes (version of 10 November 2026); e_i denotes the i-th unit vector and $\mathbf{1} = (1, \dots, 1)^\top$.

Exercise 1 (Gershgorin discs and Rayleigh quotients) — 3 points

Let

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 6 & 1 \\ 0 & 1 & 10 \end{pmatrix} \in \mathbb{R}^{3\times 3}$$

with the eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

(a) [1 P] Determine the Gershgorin discs of A (Theorem 5.2) and explain why all eigenvalues of A lie in $[3, 8] \cup [9, 11]$. Use Corollary 5.3 to show that A is SPD and that $\operatorname{cond}_2(A) \leq \frac{11}{3}$.

(b) [2 P] Compute the Rayleigh quotients $r_A(e_1)$ and $r_A(e_3)$ (Definition 5.5). Combine Theorem 5.6(i), part (a) and $\operatorname{tr} A = \lambda_1 + \lambda_2 + \lambda_3$ (Section 5.1, (f)) to show

$$\lambda_3 \in [3, 4], \qquad \lambda_2 \in [5, 7], \qquad \lambda_1 \in [10, 11].$$

Conclude that exactly one eigenvalue of A lies in $[9, 11]$, and that $\frac{5}{2} \leq \operatorname{cond}_2(A) \leq \frac{11}{3}$.

Exercise 2 (Power method and inverse iteration by hand) — 3 points

Let

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}.$$

(a) [1 P] Compute the eigenvalues $\lambda_1 > \lambda_2$ and eigenvectors of A. Carry out two steps of the power method (Algorithm 5.7) with $x^{(0)} = e_1$ and compute $\mu^{(1)}$ and $\mu^{(2)}$ as fractions. The normalisation may be omitted: explain why it changes neither the directions of the iterates nor the Rayleigh quotients.

(b) [1 P] Expand e_1 in the eigenvectors and show that

$$A^k e_1 = \frac{2}{5}\, 6^k \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \lambda_1 - \mu^{(k)} = \frac{5}{4 \cdot 36^k + 1} \qquad (k \geq 1).$$

Deduce $\mu^{(3)}$ without a further step. Compare with Theorem 5.8: what is q, and what is the asymptotic factor by which the error of $\mu^{(k)}$ decreases per step?

(c) [1 P] Which eigenvalue of A does inverse iteration with the shift $\mu = 2$ (Algorithm 5.11) approximate? With which rates do the directions of the iterates and the Rayleigh quotients converge (Proposition 5.10 and Theorem 5.8)? Carry out two steps from $x^{(0)} = e_1$ (normalisation may again be omitted) and compute the two Rayleigh quotients.

Exercise 3 (PageRank) 3 points

(a) [1 P] Three pages have the links $1 \to 2$, $1 \to 3$ and $2 \to 3$; page 3 has no outgoing links. Set up the column-stochastic matrix A of Section 5.6 (note the dangling page) and compute the vector $\pi \geq 0$ with $\|\pi\|_1 = 1$ and $A\pi = \pi$. Which ranking of the pages results?

(b) [1 P] Let $G \in \mathbb{R}^{N\times N}$, $N \geq 2$, be column-stochastic with all entries positive, for instance $G = (1-d)A + \frac{d}{N}\mathbf{1}\mathbf{1}^\top$ with $d \in (0,1)$ as in Theorem 5.13. Show with Gershgorin's theorem for the columns (Theorem 5.2 applied to $G^\top$) that every eigenvalue $\lambda \neq 1$ of G satisfies $|\lambda| < 1$. (Hint: for $0 < c < 1$, $|z - c| \leq 1 - c$ and $|z| = 1$ imply $\operatorname{Re} z \geq 1$.) Why does the same argument fail to exclude eigenvalues $\lambda \neq 1$ with $|\lambda| = 1$ for $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$?

(c) [1 P] Two pages linking to each other give $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. For $d \in (0,1)$ compute the matrix $G = (1-d)A + \frac{d}{2}\mathbf{1}\mathbf{1}^\top$, its eigenvalues with eigenvectors, and π. Show that the iterates $x^{(k+1)} := Gx^{(k)}$ with $x^{(0)} = e_1$ satisfy $\|x^{(k)} - \pi\|_1 = (1-d)^k$, so that the first inequality of Theorem 5.13(iii) holds with equality.

Exercise 4 (Lagrange interpolation and the error formula) 3 points

Let $f(x) = \dfrac{1}{1+x}$ on $[0,2]$, and let $p \in \mathcal{P}_2$ be the interpolation polynomial of f at the nodes $0, 1, 2$, with the node polynomial $\omega(x) = x(x-1)(x-2)$.

(a) [1 P] Compute the Lagrange basis polynomials L_0, L_1, L_2 and p in Lagrange form (Theorem 6.1), and write p in the monomial basis. Check the interpolation conditions.

(b) [1 P] Compute the error $f(\frac{1}{2}) - p(\frac{1}{2})$ and determine the intermediate point $\xi \in (0,2)$ of Theorem 6.2 for $x = \frac{1}{2}$ explicitly.

(c) [1 P] Bound $\max_{[0,2]} |f - p|$ with Theorem 6.2, once with the exact value of $\max_{[0,2]} |\omega|$ and once with Lemma 6.3. The actual maximal error is about 0.046 (computed numerically). Which estimate is mainly responsible for the overestimation? Compare $|f'''(\xi)|$ from (b) with $\max_{[0,2]} |f'''|$.

Exercise 5 (Chebyshev polynomials and nodes) 3 points

(a) [1 P] Compute T_2 and T_3 with the recursion of Lemma 6.6(i), and the Chebyshev nodes for $n = 2$ (Theorem 6.7). Determine the node polynomials ω of these nodes and of the equidistant nodes $-1, 0, 1$, and their maxima on $[-1,1]$. (For the equidistant nodes you may reuse Exercise 4(c) after the change of variable $s = x - 1$.)

(b) [2 P] Prove Theorem 6.7 for $n = 1$ directly: every monic polynomial $q(x) = x^2 + bx + c$ with $b, c \in \mathbb{R}$ satisfies $\max_{x\in[-1,1]} |q(x)| \geq \frac{1}{2}$, with equality only for $q(x) = x^2 - \frac{1}{2} = \frac{1}{2}T_2(x)$. (Hint: compute $q(1) - 2q(0) + q(-1)$.) Which two nodes in $[-1,1]$ therefore minimise the node factor $\max_{[-1,1]} |\omega|$ in the error bound of Theorem 6.2 for linear interpolation, and how does the resulting bound compare with (6.1) for the nodes -1 and 1?