Exercise Sheet 3

The singular value decomposition (Chapter 4)

Handed out Thursday 5 November 2026 · discussed in the tutorials of Friday 13 November 2026

15 points, about two hours

Result numbers refer to the lecture notes; $\|\cdot\|_2$ is the Euclidean norm and the induced matrix norm, $\|\cdot\|_F$ the Frobenius norm.

Exercise 1 (An SVD by hand) — 5 points

Let

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 2}.$$

(a) [3 P] Compute a singular value decomposition $A = U\Sigma V^\top$ following the proof of Theorem 4.1: the eigenvalues of $A^\top A$ and an orthonormal basis v_1, v_2 of eigenvectors, the singular values $\sigma_1 \geq \sigma_2$, the vectors $u_i := Av_i/\sigma_i$, and a third vector u_3 that completes u_1, u_2 to an orthonormal basis of $\mathbb{R}^3$ (it is orthogonal to u_1 and u_2: solve the two linear equations, or use the cross product). Write down U, Σ, V and the reduced SVD (4.1), and check that $U\Sigma V^\top = A$.

(b) [1 P] Read off from (a), naming the result you use: $\operatorname{rank} A$, $\|A\|_2$, $\|A\|_F$, $\operatorname{cond}_2(A)$ (Definition 3.6), an orthonormal basis of $\operatorname{Im} A$ and one of $\ker A^\top$.

(c) [1 P] Determine the truncated sum A_1 of (4.2), which by Theorem 4.10 is a best rank-one approximation of A, and the errors $\|A - A_1\|_2$ and $\|A - A_1\|_F$. Verify that $A - A_1 = \sigma_2 u_2 v_2^\top$.

Exercise 2 (Reading off the SVD) — 4 points

Let $A = U\Sigma V^\top$ be a singular value decomposition of $A \in \mathbb{R}^{m\times n}$ with singular values $\sigma_1 \geq \cdots \geq \sigma_p \geq 0$, $p = \min(m, n)$, and let $r = \operatorname{rank} A$.

(a) [1 P] Write down a singular value decomposition of $A^\top$ and one of QAP, where $Q \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{n\times n}$ are orthogonal (check that the factors have the properties required in Theorem 4.1). Conclude that $A^\top$, QAP and A have the same singular values, and hence $\|A^\top\|_2 = \|A\|_2$, $\|QAP\|_2 = \|A\|_2$ and $\|QAP\|_F = \|A\|_F$ (compare Proposition 1.23(i),(ii)).

(b) [2 P] Prove that

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{r}\,\|A\|_2,$$

and determine for which matrices A equality holds on the left and for which on the right (characterise them by their singular values). The left inequality was shown on Sheet 1 without the SVD.

(c) [1 P] Let $m = n$ and A regular. Show with Proposition 4.3(v): if $\Delta A \in \mathbb{R}^{n\times n}$ satisfies $\|\Delta A\|_2 < \sigma_n$, then $A + \Delta A$ is regular. Compare with the hypothesis $\|A^{-1}\|\,\|\Delta A\| < 1$ of Theorem 2.5: rewrite it in the 2-norm using Proposition 4.3(iii). What does Proposition 4.3(v) add: give a ΔA with $\|\Delta A\|_2 = \sigma_n$ for which $A + \Delta A$ is singular.

Exercise 3 (The pseudoinverse) 4 points

Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 4 \end{pmatrix} \in \mathbb{R}^{3\times 2}, \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) [1 P] Determine $\operatorname{rank} A$, the reduced SVD $A = \sigma_1 u_1 v_1^\top$ (as in Example 4.6: eigenvalues and eigenvectors of the 2×2 matrix $A^\top A$) and $\ker A$.

(b) [2 P] Compute the pseudoinverse A^+ (Definition 4.4) and $x^+ := A^+ b$. Show with the normal equations (Theorem 3.4) that the least squares solutions of $Ax = b$ are exactly the points of the line $x_1 + 2x_2 = \frac{5}{9}$, that x^+ is the point of this line with the smallest Euclidean norm (Theorem 4.5), and that $x^+ \perp \ker A$ (compare Example 4.6).

(c) [1 P] Compute AA^+ and A^+A and verify Theorem 4.5(iii): both matrices are symmetric and idempotent, the image of AA^+ is $\operatorname{Im} A$, and the image of A^+A is $(\ker A)^\perp$. Compute AA^+b and the minimal residual norm $\|b - AA^+b\|_2 = \|b - Ax^+\|_2$.

Exercise 4 (Low-rank approximation) 2 points

(a) [1 P] Let $A = U\Sigma V^\top \in \mathbb{R}^{m\times n}$ with $\operatorname{rank} A = r$, let $1 \le k < r$, and let A_k be the truncated sum (4.2). Show that

$$A_k = U_k U_k^\top A = A V_k V_k^\top,$$

where $U_k \in \mathbb{R}^{m\times k}$ and $V_k \in \mathbb{R}^{n\times k}$ consist of the first k columns of U and V. (Write A as in (4.1) and compute $U_k^\top u_i$ for $i \le k$ and for $i > k$.) What does this say about the columns of A_k? Compare Theorem 4.5(iii).

(b) [1 P] Theorem 4.10 states that A_k attains the minimum of $\|A - B\|_2$ over all B of rank at most k. Show by an example with a regular $A \in \mathbb{R}^{2\times 2}$ (so that $r = 2$) and $k = 1$ that the minimiser need not be unique: find two different matrices B_1, B_2 of rank one with $\|A - B_1\|_2 = \|A - B_2\|_2 = \sigma_2$. Does this contradict the theorem?