Basics in Applied Mathematics

Part I: Numerics — Winter term 2026/27

Exercise Sheet 2

Perturbation of linear systems and pivoting (Chapter 2);
least squares problems and the QR decomposition (Chapter 3)

20 points in total; the points of a part reflect its difficulty and the expected working time.
Unless stated otherwise, all calculations are to be carried out in exact arithmetic; decimal values serve only for interpretation.

Exercise 1 (Perturbation of the matrix). [5 points]

Let

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \quad \text{(the } 2 \times 2 \text{ Hilbert matrix, Example 2.7)}, \qquad b = \begin{pmatrix} \frac{3}{2} \\ \frac{5}{6} \end{pmatrix},$$

so that $x = (1, 1)^\top$ solves $Ax = b$.

(a) Compute A^{-1} and $\operatorname{cond}_\infty(A)$ (Definition 2.1, Theorem 1.22). (1 point)

(b) Suppose that the entry $a_{22} = \frac{1}{3}$ is stored as 0.333 (rounded to three decimals) while all other data are exact, that is, the system $(A + \Delta A)\tilde{x} = b$ with $\Delta A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -1/3000 \end{smallmatrix}\right)$ is solved instead of $Ax = b$. Compute $\tilde{x}$ exactly and the relative error $\|\tilde{x} - x\|_\infty / \|x\|_\infty$. Check the hypothesis of Theorem 2.5 and compare the relative error with the bound of the theorem. If every entry of A and of b were rounded to three decimals (perturbations of modulus at most $5 \cdot 10^{-4}$), which relative error of the solution would Theorem 2.5 guarantee at most? (2 points)

(c) Show that the hypothesis $\|A^{-1}\| \, \|\Delta A\| < 1$ of Theorem 2.5 cannot be dropped: find $\Delta A \in \mathbb{R}^{2\times 2}$ with $\|A^{-1}\|_\infty \|\Delta A\|_\infty = 1$ such that $A + \Delta A$ is singular. (Hint: take $y \in \mathbb{R}^2$ with $\|y\|_\infty = 1$ and $\|A^{-1}y\|_\infty = \|A^{-1}\|_\infty$, as in the proof of Theorem 1.22, put $\hat{x} := A^{-1}y$, and look for a matrix ΔA of rank one with $(A + \Delta A)\hat{x} = 0$.) Explain why the construction works for every regular matrix, and conclude with Lemma 2.4 that the distance $\min\{\|\Delta A\|_\infty : A + \Delta A \text{ singular}\}$ from a regular matrix A to the nearest singular matrix equals $1/\|A^{-1}\|_\infty = \|A\|_\infty / \operatorname{cond}_\infty(A)$. (2 points)

Exercise 2 (Pivoting). [5 points]

(a) Consider the system $Ax = b$,

$$\begin{pmatrix} 10^{-4} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Carry out Gaussian elimination and the backward substitution (Algorithms 2.10 and 2.9) in a decimal floating-point arithmetic with three significant digits—the result of every arithmetic operation is rounded to three significant digits—once without and once with the exchange of the two rows (partial pivoting). Compare both computed solutions with the exact solution, determine the product $\tilde{L}\tilde{U}$ of the factors computed without exchange, and explain with Example 2.15 where the accuracy is lost and why this is a property of the algorithm and not of the problem (compute $\operatorname{cond}_\infty(A)$). (2 points)

(b) Compute the decomposition $PA = LU$ with partial pivoting (Theorem 2.16) of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{pmatrix}$$

of Sheet 1, Exercise 4(a): give P, L and U, check $LU = PA$, solve $Ax = b$ for $b = (4, 10, 24)^\top$ via $LUx = Pb$, and compute $\det A$ from U and P. Compare the multipliers with those of the decomposition without row exchanges from Sheet 1. (2 points)

(c) Let $A \in \mathbb{R}^{n\times n}$ be SPD. Show that Gaussian elimination without row exchanges does not break down and that all pivots are positive: $u_{kk} = \det A_k / \det A_{k-1} > 0$ (equation (2.3), Proposition 2.19). Show moreover that $u_{kk} = l_{kk}^2$, where $L = (l_{ij})$ is the Cholesky factor of A (hint: write $A = LL^\top = (LD^{-1})(DL^\top)$ with $D = \operatorname{diag}(l_{11}, \ldots, l_{nn})$ and use the uniqueness in Theorem 2.13), and check both statements for the matrix C of Sheet 1, Exercise 4(b). (1 point)

Exercise 3 (Least squares and the normal equations). [5 points]

At the times $t = 0, 1, 2, 3$ the values $y = 2, 2, 4, 8$ of a quantity were measured, and a linear law $y(t) = x_1 + x_2 t$ is assumed.

(a) Write the determination of x_1, x_2 as a least squares problem $\|Ax - b\|_2 \to \min$ (Definition 3.1), set up the normal equations (Theorem 3.4) and solve them. Compute the projection $p = Ax^*$ and the residual $r = b - p$, check $A^\top r = 0$, and verify $\|b\|_2^2 = \|p\|_2^2 + \|r\|_2^2$. Why is the least squares solution unique here? (2 points)

(b) Compute $\operatorname{cond}_2(A)$ from the eigenvalues of $A^\top A$ (Definition 3.6) and $\operatorname{cond}_2(A^\top A)$. Suppose the same values had been measured at the times $t = 100, 101, 102, 103$: compute $A^\top A$ and $\operatorname{cond}_2(A)$ for these nodes (an approximation based on $\lambda_{\min}\lambda_{\max} = \det(A^\top A)$ and $\lambda_{\min} + \lambda_{\max} = \operatorname{trace}(A^\top A)$ suffices). Which difficulty does Proposition 3.7 predict for the normal equations in this case, and how can it be avoided without changing the fitted line? (1 point)

(c) Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and let x^* satisfy the normal equations $A^\top A x^* = A^\top b$. Prove that

$$\|Ax - b\|_2^2 = \|A(x - x^*)\|_2^2 + \|b - Ax^*\|_2^2 \qquad \text{for all } x \in \mathbb{R}^n,$$

and deduce that x^* is a least squares solution and that the least squares solution is unique if and only if $\ker A = \{0\}$. Determine all least squares solutions of $Ax = b$ for $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $b = (1, 2, 3)^\top$, together with the corresponding projections and residuals. (2 points)

Exercise 4 (QR decomposition by Householder reflections). [5 points]

(a) Let

$$A = \begin{pmatrix} 2 & 0 \\ 2 & 5 \\ 1 & 5 \end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}.$$

Compute the QR decomposition of A with Algorithm 3.22 by hand: give the vectors v_1, v_2 of the two reflections (Lemma 3.21), the matrix R_1 and the vector $Q^\top b$, obtained by applying the reflections to b (the matrix Q is not needed). Solve the least squares problem $\|Ax - b\|_2 \to \min$ with Theorem 3.13 and read off the minimal residual. Check the result with the normal equations and with the residual $r = b - Ax^*$, and verify $R_1^\top R_1 = A^\top A$ (Remark 3.14). (3 points)

(b) Let $x \in \mathbb{R}^m \setminus \{0\}$ be no multiple of e_1. Show that there are exactly two Householder reflections H_v with $H_v x \in \operatorname{span}\{e_1\}$, namely those with $v \in \operatorname{span}\{x - \|x\|_2 e_1\}$ and with $v \in \operatorname{span}\{x + \|x\|_2 e_1\}$, and that they map x to $\|x\|_2 e_1$ and to $-\|x\|_2 e_1$, respectively. (Hint: Lemma 3.20 and the computation in the proof of Lemma 3.21.) Which of the two does Lemma 3.21 choose, and why? Illustrate the difference for $x = (1, 10^{-4})^\top$ in double precision (compare Example 1.5). (2 points)