Basics in Applied Mathematics

Part I: Numerics — Winter term 2026/27

Exercise Sheet 1

Floating-point arithmetic, conditioning and norms (Chapter 1);
linear systems, LU and Cholesky decomposition (Chapter 2)

20 points in total; the points of a part reflect its difficulty and the expected working time.
Unless stated otherwise, all calculations are to be carried out in exact arithmetic; decimal values serve only for interpretation.

Exercise 1 (Condition numbers and stability). [2 points]

Compute the relative condition number $\kappa_{\mathrm{rel}}(x)$ (Definition 1.6, Lemma 1.7) of (i) $f(x) = \ln x$ for $x > 0$, $x \neq 1$, and (ii) $f(x) = x^p$ for a fixed $p \in \mathbb{R}$ and $x > 0$. Where is each problem ill-conditioned? Give a one-sentence interpretation of your answers for (i) and for (ii) with $p = -1$.

Exercise 2 (Vector and matrix norms). [7 points]

(a) Let $x = (1, -2, 2)^\top \in \mathbb{R}^3$. Compute $\|x\|_1$, $\|x\|_2$, $\|x\|_\infty$ and check the six inequalities of Theorem 1.16 (with $n = 3$). Prove that $\|x\|_2^2 \leq \|x\|_1 \, \|x\|_\infty$ for all $x \in \mathbb{R}^n$, and determine all vectors for which equality holds. (2 points)

(b) Let

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}.$$

Compute $\|A\|_1$, $\|A\|_\infty$, $\|A\|_F$ and $\|A\|_2$ (Section 1.4.2, Theorem 1.22). Give a vector x with $\|x\|_\infty = 1$ and $\|Ax\|_\infty = \|A\|_\infty$, and a vector y with $\|y\|_1 = 1$ and $\|Ay\|_1 = \|A\|_1$. (2 points)

(c) Prove that every $A \in \mathbb{R}^{m \times n}$ satisfies

$$\text{(i)} \quad \|A\|_2 \leq \|A\|_F \qquad \text{and} \qquad \text{(ii)} \quad \|A\|_2^2 \leq \|A\|_1 \, \|A\|_\infty.$$

For (i) use the Cauchy–Schwarz inequality and Definition 1.19; for (ii) combine Theorem 1.22 with Propositions 1.20(iii), 1.23(iii) and 1.24. Check both inequalities for the matrix of (b). (3 points)

Exercise 3 (Condition number of a linear system). [7 points]

Let $\varepsilon = 10^{-4}$ and

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1+\varepsilon \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 2+\varepsilon \end{pmatrix},$$

so that $x = (1, 1)^\top$ solves $Ax = b$.

(a) Compute A^{-1} and $\mathrm{cond}_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$. Solve $A(x + \Delta x) = b + \Delta b$ for $\Delta b = (0, \varepsilon)^\top$ and compare $\|\Delta x\|_\infty / \|x\|_\infty$ with $\|\Delta b\|_\infty / \|b\|_\infty$. How far is the bound of Theorem 2.2 from being attained? (2 points)

(b) Consider the candidate solution $\tilde{x} = (2, 0)^\top$. Compute the residual $r = b - A\tilde{x}$ and the error $x - \tilde{x}$; compare the relative residual $\|r\|_\infty / \|b\|_\infty$ with the relative error $\|x - \tilde{x}\|_\infty / \|x\|_\infty$, and check the two bounds

$$\frac{1}{\mathrm{cond}(A)} \, \frac{\|r\|}{\|b\|} \;\leq\; \frac{\|x - \tilde{x}\|}{\|x\|} \;\leq\; \mathrm{cond}(A) \, \frac{\|r\|}{\|b\|}$$

of Proposition 2.6(iii). What does the example teach about judging a computed solution by its residual? (2 points)

(c) Show directly, with the two inequalities used in the proof of Theorem 2.2, that under its assumptions also

$$\frac{\|\Delta x\|}{\|x\|} \;\geq\; \frac{1}{\operatorname{cond}(A)}\,\frac{\|\Delta b\|}{\|b\|} \qquad (\Delta b \neq 0)$$

(this is the lower bound of Proposition 2.6(iii) for $\tilde{x} = x + \Delta x$), and interpret the two bounds together, for well-conditioned and for ill-conditioned A. For the matrix of (a), which values of the amplification factor $\big(\|\Delta x\|_\infty/\|x\|_\infty\big)/\big(\|\Delta b\|_\infty/\|b\|_\infty\big)$ actually occur? (Consider also $\Delta b = t\,b$.) (3 points)

Exercise 4 (LU and Cholesky decomposition by hand). [4 points]

(a) Compute the LU decomposition (without row exchanges) of

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{pmatrix},$$

and verify $LU = A$. Use it to solve $Ax = b$ for $b = (4, 10, 24)^\top$ and for $b' = (2, 4, 10)^\top$ by forward and backward substitution (Algorithm 2.9), and read off $\det A$. Which part of the computation has to be repeated for b', and which part not (Section 2.4.2)? (2 points)

(b) Compute the Cholesky factor L of

$$C = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 2 & -3 \\ 2 & -3 & 14 \end{pmatrix}$$

with the column-by-column algorithm of Section 2.5 (Algorithm 2.21), check $LL^\top = C$ and compute $\det C$ from L. Apply the same algorithm to

$$D = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

and state what you conclude (Theorem 2.20, Remark 2.22(iii)). (2 points)