Basics in Applied Mathematics – Part I: Numerics

Lecture 10: Gaussian quadrature (end of Chapter 7) — ordinary differential equations (Chapter 8)

JProf. Dr. Diyora Salimova

Department of Applied Mathematics, University of Freiburg

19 November 2026

Recap of Lecture 9: towards Gaussian quadrature

Quadrature rule $Q(f) = \sum_{i=0}^{n} w_i\, f(x_i)$ for $I(f) = \int_a^b f(x)\,\mathrm{d}x$, with $n+1$ distinct nodes:

- degree of exactness $\leq 2n+1$: $Q(\omega^2) = 0 < I(\omega^2)$, $\omega(x) = \prod_i (x - x_i)$ (Definition 7.1);
- interpolatory rule, $w_i = \int_a^b L_i(x)\,\mathrm{d}x$: exact of degree at least n (Lemma 7.2);
- Legendre polynomials on $[-1, 1]$: $\deg P_k = k$, $P_k \perp \mathcal{P}_{k-1}$ (Definition 7.6); P_{n+1} has $n+1$ distinct real roots, all in $(-1, 1)$ (Lemma 7.7).

Today: why the roots of P_{n+1} are the best nodes (the proof of Theorem 7.8); then Chapter 8, ordinary differential equations — the last chapter of Part I.

Theorem 7.8 (Gauss–Legendre quadrature)

Let $x_0, \dots, x_n$ be the roots of P_{n+1} and let $w_i = \int_{-1}^{1} L_i(x)\,\mathrm{d}x$ be the weights of the interpolatory rule with these nodes. Then:

(i) the rule $Q(f) = \sum_{i=0}^{n} w_i f(x_i)$ is exact of degree $2n+1$;

(ii) all weights are positive, and $\sum_i w_i = 2$;

(iii) no quadrature rule with $n+1$ nodes in $[-1,1]$ is exact of degree $2n+2$.

Blackboard: proof of (i) and (ii); (iii): ω^2 as in Definition 7.1.

$n+1$ function values, exact of degree $2n+1$: the largest possible degree.

Summary of Chapter 7

- **Quadrature rule** $Q(f) = \sum_i w_i f(x_i)$; interpolatory rule: $w_i = \int L_i$, exact of degree $\geq n$; degree of exactness $\leq 2n+1$; symmetric rules gain one degree.
- **Local errors:** midpoint $\frac{h^3}{24} f''(\xi)$, trapezoidal $-\frac{h^3}{12} f''(\xi)$. **Composite trapezoidal rule:** $-\frac{b-a}{12} h^2 f''(\xi)$.
- **EOC** checks the order; for non-smooth f (e.g. $x^{1/3}$) the order drops on a uniform mesh, also for Gauss rules.
- **Gauss–Legendre:** nodes = roots of P_{n+1}, exact of degree $2n+1$ (optimal), positive weights, convergence for every continuous f.

Definition 8.1 (initial value problem)

Let $t_0 < T$, $d \in \mathbb{N}$, let $f\colon [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous, and let $y_0 \in \mathbb{R}^d$. The **initial value problem (IVP)** is to find $y \in C^1([t_0, T]; \mathbb{R}^d)$ with

$$y'(t) = f(t, y(t)) \quad \text{for all } t \in [t_0, T], \qquad y(t_0) = y_0.$$

- $y' = \lambda y$: $y(t) = y_0 e^{\lambda(t - t_0)}$; the logistic equation $y' = y(1 - y)$.
- Pendulum $\theta'' = -\sin\theta$: $y = (\theta, \theta')$ solves the first-order system $y_1' = y_2$, $y_2' = -\sin y_1$.
- Predator–prey, epidemics, N bodies ($d = 6N$), the heat equation discretised in space.

Explicit solution formulas exist only in special cases $\Longrightarrow$ numerical methods.

Example: the logistic equation

$y' = y(1-y)$, $y(0) = y_0$: population growth with capacity 1; here $f(t,y) = y(1-y)$.

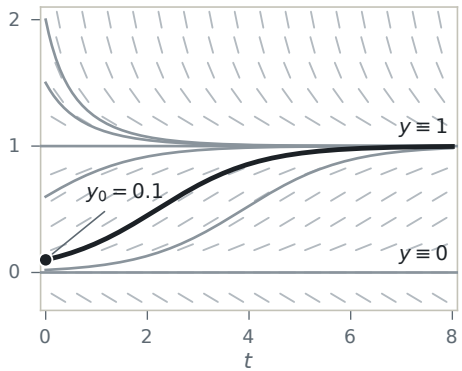
Slope field: at each point (t,y) a short segment of slope $f(t,y)$. A solution is tangent to these segments.

$y_0 = 0.1$**: what does the solution do?**

It increases and tends to 1:

$$y(t) = \frac{y_0 e^t}{1 - y_0 + y_0 e^t} = \frac{e^t}{9 + e^t}.$$

One initial value, one solution; $y \equiv 0$ and $y \equiv 1$ are constant solutions.



Theorem 8.3 (Picard–Lindelöf)

Let f be continuous and **globally Lipschitz continuous in** y: there is $L > 0$ with

$$\|f(t,y) - f(t,z)\| \le L\|y - z\| \qquad \text{for all } t \in [t_0, T],\ y, z \in \mathbb{R}^d.$$

Then the IVP has exactly one solution $y \in C^1([t_0, T]; \mathbb{R}^d)$ for every $y_0 \in \mathbb{R}^d$, and the solutions y, z with initial values y_0, z_0 satisfy

$$\|y(t) - z(t)\| \le e^{L(t-t_0)}\, \|y_0 - z_0\| \qquad \text{for all } t \in [t_0, T].$$

Globally Lipschitz in y? (a) $y' = -2y$; (b) the pendulum; (c) $y' = y(1-y)$.

(a) $L = 2$. (b) $L = 1$, since $|\sin a - \sin b| \le |a - b|$.

(c) No: $|f(y) - f(z)| = |1 - y - z|\,|y - z|$; $y_0 = -1$: $y(t) = \frac{-e^t}{2 - e^t}$ blows up at $t = \ln 2$.

Grid $t_n := t_0 + nh$, $n = 0, \dots, N$, step size $h := \frac{T - t_0}{N}$; approximations $y_n \approx y(t_n)$.

Integrate the ODE over one step: $y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s))\,\mathrm{d}s$.

Which quadrature rule uses only the value at t_n? The left rectangle rule: $h\, f(t_n, y(t_n))$.

Algorithm 8.5 (explicit Euler method)

Set y_0 to the initial value. For $n = 0, 1, \dots, N-1$ compute $y_{n+1} := y_n + h\, f(t_n, y_n)$.

The same method from the tangent: $y(t_{n+1}) = y(t_n) + h\, y'(t_n) + O(h^2)$.
Right rectangle rule: the **implicit** Euler method $y_{n+1} = y_n + h\, f(t_{n+1}, y_{n+1})$.

Example: $y' = -2y, \quad y(0) = 1$

Exact solution $y(t) = e^{-2t}$; Euler on $[0, 3]$ with $h = 0.6$.

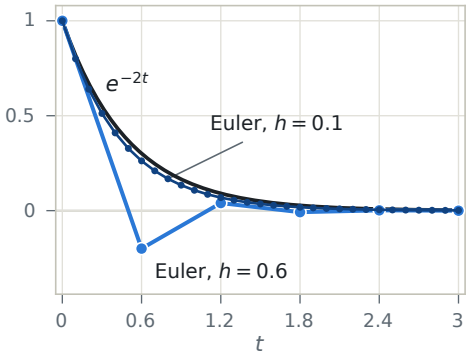
$y_1 =$ **?** $\quad y_2 =$ **?**

$y_1 = 1 + 0.6 \cdot (-2) = -0.2, \quad y_2 = 0.04.$
Every step multiplies by $1 - 2h = -0.2$: the signs alternate, although $e^{-2t} > 0$.

Maximal error over the grid:

h	0.6	0.1	0.05
error	0.50	0.040	0.019

Halving h about halves the error.



Definition 8.6 (one-step method, local error, consistency, convergence)

A **one-step method** has the form $y_{n+1} = y_n + h\,\Phi(t_n, y_n; h),\ \ n = 0, \dots, N-1$ (Euler: $\Phi(t, y; h) = f(t, y)$). Let y be the exact solution and $h = \frac{T - t_0}{N} \le h_0$.

- **local error** $\tau_n(h) := \dfrac{y(t_{n+1}) - y(t_n)}{h} - \Phi\big(t_n, y(t_n); h\big)$;
  consistent of order p if $\max_{0 \le n < N} \|\tau_n(h)\| \le C h^p$;
- **global error** $e_n := y_n - y(t_n)$; **converges with order** p if $\max_{0 \le n \le N} \|e_n\| \le C h^p$.

The constants C are independent of h and n.

Consistency: the exact solution *almost* satisfies the scheme. Convergence: y_n is close to $y(t_n)$. One step from $y(t_n)$ errs by $h\tau_n(h) = O(h^{p+1})$; $N = \frac{T - t_0}{h}$ steps give $O(h^p)$ — **if** the errors are not amplified too much.

Lemma 8.7 (consistency of Euler's method)

Let the solution satisfy $y \in C^2([t_0, T]; \mathbb{R}^d)$. Then for Euler's method

$$\tau_n(h) = \frac{1}{h} \int_{t_n}^{t_{n+1}} (t_{n+1} - s)\, y''(s)\, \mathrm{d}s, \qquad \text{hence} \qquad \|\tau_n(h)\| \le \frac{h}{2} \max_{[t_0,T]} \|y''\| :$$

Euler's method is consistent of order 1.

Blackboard: proof.

Example $y' = -2y$: $\tau_n(h) = \dfrac{e^{-2h} - 1 + 2h}{h}\, y(t_n) = \big(2h + O(h^2)\big)\, y(t_n)$: order 1, not 2.

Lemma 8.8 (discrete Gronwall lemma)

Let $L, h > 0$, $b \geq 0$, and let $a_0, \ldots, a_N \geq 0$ satisfy $a_{n+1} \leq (1 + hL)a_n + b$ for $n = 0, \ldots, N-1$. Then $a_n \leq e^{nhL} a_0 + \frac{e^{nhL}-1}{hL}\, b$ for $n = 0, \ldots, N$.

Theorem 8.9 (convergence of Euler's method)

Let f be continuous and globally Lipschitz continuous in y with constant L, and let $y \in C^2([t_0, T]; \mathbb{R}^d)$. Then Euler's method converges with order 1:

$$\max_{0 \leq n \leq N} \|y_n - y(t_n)\| \leq \frac{e^{L(T-t_0)} - 1}{L} \cdot \frac{h}{2} \max_{[t_0,T]} \|y''\|.$$

Blackboard: proof of Theorem 8.9 (Lemma 8.8: induction, in the notes).

Consistency + Lipschitz condition $\Longrightarrow$ convergence — for every one-step method.

Higher order: better quadrature rules

Euler: the left rectangle rule for $\int_{t_n}^{t_{n+1}} f(s, y(s))\,\mathrm{d}s$.
Better rules: **predict** y by Euler steps.

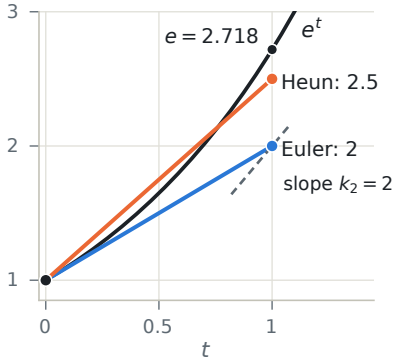
Heun's method (trapezoidal rule):

$$k_1 = f(t_n, y_n), \quad k_2 = f(t_n + h,\, y_n + hk_1),$$
$$y_{n+1} = y_n + \tfrac{h}{2}(k_1 + k_2).$$

Explicit midpoint method (midpoint rule):

$$k_1 = f(t_n, y_n), \quad k_2 = f\big(t_n + \tfrac{h}{2},\, y_n + \tfrac{h}{2}k_1\big),$$
$$y_{n+1} = y_n + hk_2.$$

Both have order 2, with two evaluations of f per step.



$y' = y$, $y_0 = 1$, $h = 1$:
one Heun step?

$$k_1 = 1, \quad k_2 = f(1, 2) = 2,$$
$$y_1 = 2.5.$$

RK4: four stages, order 4,

$$k_1 = f(t_n, y_n), \quad k_2 = f\big(t_n + \tfrac{h}{2},\, y_n + \tfrac{h}{2}k_1\big), \quad k_3 = f\big(t_n + \tfrac{h}{2},\, y_n + \tfrac{h}{2}k_2\big),$$

$$k_4 = f(t_n + h,\, y_n + hk_3), \qquad y_{n+1} = y_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

If f does not depend on y: $y_{n+1} - y_n = \tfrac{h}{6}\big(f(t_n) + 4f(t_n + \tfrac{h}{2}) + f(t_{n+1})\big)$, a quadrature rule.

Explicit s-stage Runge–Kutta method with coefficients a_{ij} $(1 \le j < i \le s)$, b_i, c_i:

$$k_i = f\Big(t_n + c_i h,\, y_n + h\sum_{j=1}^{i-1} a_{ij}k_j\Big) \quad (i = 1, \dots, s), \qquad y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i,$$

a one-step method with $\Phi(t, y; h) = \sum_i b_i k_i$.
Heun: $s = 2$, $c = (0, 1)$, $a_{21} = 1$, $b = (\tfrac{1}{2}, \tfrac{1}{2})$.

Experiment: orders 1, 2 and 4

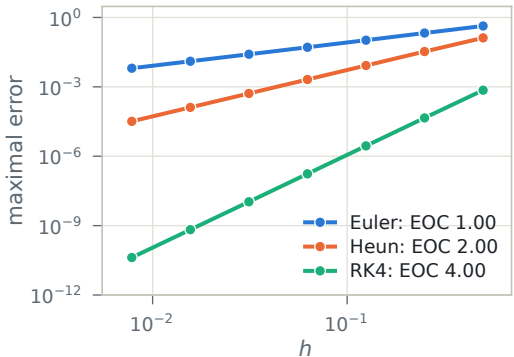
$y' = \cos(t)\, y$, $y(0) = 1$, on $[0, 5]$;
exact solution $y(t) = e^{\sin t}$.

Maximal error over the grid,
$N = 10, 20, \ldots, 640$.

Euler: EOC 1. Heun and RK4?

EOC 2.00 and 4.00.

The same cost, 80 evaluations of f:
Euler, $N = 80$: error $5.1 \cdot 10^{-2}$;
RK4, $N = 20$: error $4.5 \cdot 10^{-5}$.



Summary of Chapter 8

- **IVP** $y' = f(t, y)$, $y(t_0) = y_0$: f globally Lipschitz in y $\implies$ exactly one solution; perturbations of y_0 grow at most like $e^{L(t-t_0)}$.
- **One-step methods** $y_{n+1} = y_n + h\Phi(t_n, y_n; h)$: local error $\tau_n(h)$, consistency and convergence of order p.
- **Euler's method** (left rectangle rule): consistent of order 1 + Lipschitz condition $\implies$ convergence with order 1, constant with $e^{L(T-t_0)}$.
- **Runge–Kutta methods** (better quadrature rules, predicted stage values): Heun order 2, RK4 order 4.

Part I at a glance

Chapters 1–8

1 Floating-point numbers, conditioning, stability, norms — the language of every chapter

Matrices: numerical linear algebra

2 Linear systems
$\mathrm{cond}(A)$, LU with pivoting, Cholesky

→

3 Least squares
normal equations, QR, Householder

→

4 SVD
pseudoinverse, best low-rank approximation

→

5 Eigenvalues
Gershgorin, power method, PageRank

Functions: approximation and discretisation

6 Polynomial interpolation
Lagrange form, error formula, Chebyshev nodes

→

7 Numerical integration
integrate the interpolant: composite rules, EOC, Gauss

→

8 Differential equations
quadrature over each step: Euler, Runge–Kutta

Three questions in every chapter

Conditioning
How sensitive is the problem?
κ_{rel}, $\mathrm{cond}(A)$, $e^{L(t-t_0)}$

Stability
Does the algorithm amplify errors?
pivoting, Householder, $(1+hL)^n$

Order of convergence
How fast does the error decrease?
h^p, checked with the EOC

After Part I

What comes next

- Part II, Stochastics: from Tuesday 24 November.
- Part III, Optimization: from 12 January.
- Oral exam: all three parts are prepared, two are examined.

Organisation

- Today: Sheet 5 (Chapters 7–8).
- Fri 20 Nov: tutorials (Sheet 4), HS II.
- Fri 27 Nov: tutorials (Sheet 5); **Test 2**, 13:45–14:15, Chapters 5–8.

Thank you!