Basics in Applied Mathematics – Part I: Numerics

Lecture 8: PageRank — polynomial interpolation: the Lagrange form, the interpolation error, the Runge phenomenon, Chebyshev nodes

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Theorem 5.2 (Gershgorin)

Every eigenvalue of $A \in \mathbb{C}^{n\times n}$ lies in $D_1 \cup \cdots \cup D_n$, $\quad D_i = \{z \in \mathbb{C} : \ |z - a_{ii}| \le r_i\}$, $r_i = \sum_{j\neq i} |a_{ij}|$.

Theorem 5.6 (Rayleigh quotient $r_A(x) = x^\top Ax/x^\top x$, A symmetric)

Eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$, orthonormal eigenvectors $v_1, \ldots, v_n$. Then $\lambda_n \le r_A(x) \le \lambda_1$ for all $x \neq 0$, and
if $\|x\|_2 = 1$ and $\|x - v_j\|_2 = \varepsilon$, then $\ |r_A(x) - \lambda_j| \le 2\|A\|_2\, \varepsilon^2$.

Recap of Lecture 7 (2/2): power method, inverse iteration

Sections 5.4–5.5

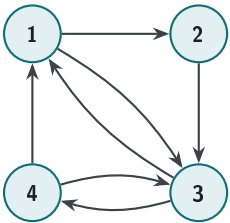
Power method (Algorithm 5.7): $x^{(k+1)} = Ax^{(k)}/\|Ax^{(k)}\|_2$. For diagonalisable A with $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$ and $x^{(0)} = \sum_i c_i v_i$, $c_1 \neq 0$ (Theorem 5.8):

- eigenvector error (up to sign) $\leq C\, q^k$, $\quad q = |\lambda_2/\lambda_1|$;
- Rayleigh quotients for symmetric A: error $\leq C' q^{2k}$.

Inverse iteration with shift μ (Algorithm 5.11): the power method for $(A - \mu I)^{-1}$, one LU decomposition;
finds the eigenvalue λ_a closest to μ, rate $|\lambda_a - \mu|/|\lambda_b - \mu|$ (λ_b: the second closest).

Today: PageRank — the power method for a matrix with billions of rows (Section 5.6); then Chapter 6: polynomial interpolation.

PageRank: the model



four pages, seven links

$$A = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

N pages; page j has n_j outgoing links. A **random surfer** on page j follows each of them with probability $1/n_j$.

Relevance x_i of page i: the stationary proportion of surfers on page i,

$$x_i = \sum_{j \to i} \frac{x_j}{n_j}, \quad \text{i.e. } x = Ax, \qquad a_{ij} = \begin{cases} 1/n_j, & j \text{ links to } i, \\ 0, & \text{otherwise.} \end{cases}$$

x is an eigenvector of A for the eigenvalue 1.

Write down A for the four pages. (Column j: the links of page j.)

Entries ≥ 0, every column sum $= 1$: A is **column-stochastic**.

Lemma 5.12

Let $A \in \mathbb{R}^{N \times N}$ be column-stochastic. Then $\|A\|_1 = 1$, every eigenvalue of A satisfies $|\lambda| \le 1$, and $\lambda = 1$ is an eigenvalue of A. For every $x \ge 0$: $Ax \ge 0$ and $\|Ax\|_1 = \|x\|_1$.

The power method needs more: **is $\lambda = 1$ always dominant?**

No. Two pages linking to each other: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has the eigenvalues ± 1; the iterates alternate between $x^{(0)}$ and $Ax^{(0)}$ (unless $x^{(0)} = (\tfrac{1}{2}, \tfrac{1}{2})^\top$).

Damping: with the **teleportation probability** $d \in (0, 1)$, typically $d \approx 0.15$, the surfer jumps to a uniformly chosen random page:

$$G := (1 - d)A + \frac{d}{N}\,\mathbf{1}\mathbf{1}^\top, \qquad \mathbf{1} = (1, \dots, 1)^\top.$$

Theorem 5.13 (PageRank)

Let $A \in \mathbb{R}^{N\times N}$ be column-stochastic, $d \in (0,1)$ and $G = (1-d)A + \frac{d}{N}\mathbf{1}\mathbf{1}^{\top}$. Then G is column-stochastic with all entries positive, and:

(i) $\|Gz\|_1 \le (1-d)\|z\|_1$ for every $z \in \mathbb{R}^N$ with $\mathbf{1}^{\top}z = 0$;

(ii) there is exactly one $\pi \in \mathbb{R}^N$ with $\pi \ge 0$, $\|\pi\|_1 = 1$ and $G\pi = \pi$; all its entries satisfy $\pi_i \ge d/N > 0$;

(iii) for every $x^{(0)} \ge 0$ with $\|x^{(0)}\|_1 = 1$ the iterates $x^{(k+1)} := Gx^{(k)}$ satisfy $\|x^{(k)} - \pi\|_1 \le (1-d)^k \|x^{(0)} - \pi\|_1 \le 2(1-d)^k$.

The error bound decreases by the factor $1-d \approx 0.85$ per step, **whatever N and the links**: a factor of ten every $\ln 10/|\ln 0.85| \approx 14.2$ steps.

G is dense and is never formed: one step costs $O(L+N)$ operations for L links — feasible for billions of pages.

Example: four pages

Power method with A itself, $x^{(0)} = \frac{1}{4}\mathbf{1}$ (no damping needed here: the other eigenvalues of A have modulus 0.5).

Which page ranks first?

k	0	1	2	3	4	5	6	7
page 1	0.25	0.25	0.3125	0.2812	0.2812	0.2891	0.2852	0.2852
page 2	0.25	0.125	0.125	0.1562	0.1406	0.1406	0.1445	0.1426
page 3	0.25	0.5	0.3125	0.4062	0.375	0.3828	0.3789	0.3828
page 4	0.25	0.125	0.25	0.1562	0.2031	0.1875	0.1914	0.1895

Limit: $\pi = \left(\frac{2}{7}, \frac{1}{7}, \frac{8}{21}, \frac{4}{21}\right)^\top \approx (0.286,\ 0.143,\ 0.381,\ 0.190)^\top$: ranking $3,\ 1,\ 4,\ 2$.

Pages 2 and 4 both have one incoming link. Why does page 4 rank higher?

Page 4 receives $\frac{1}{2}\pi_3$ from the most relevant page, page 2 receives $\frac{1}{2}\pi_1$: links from relevant pages count more.

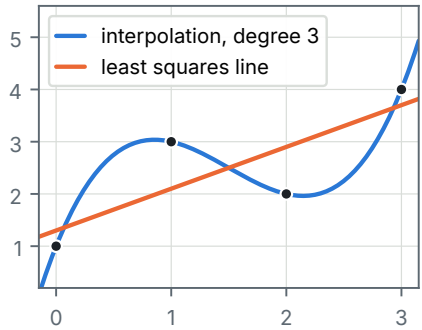
Given: $n+1$ distinct **nodes** $x_0, \dots, x_n$ and **values** $y_0, \dots, y_n$ (often $y_i = f(x_i)$).

Find: $p \in \mathcal{P}_n$ (polynomials of degree $\le n$) with $p(x_i) = y_i, \;\; i = 0, \dots, n$.

$\dim \mathcal{P}_n = n+1$: as many parameters as conditions.

Unlike least squares (Chapter 3): p passes through *all* data points.

Basis of numerical integration (Chapter 7) and of Runge–Kutta methods (Chapter 8).



Goals

(i) existence, uniqueness and the Lagrange form
(ii) the interpolation error
(iii) the Runge phenomenon and Chebyshev nodes

Which polynomial of degree n is 0 at all nodes except x_i, and 1 at x_i?

$$L_i(x) := \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} \qquad \textbf{(Lagrange basis polynomials)}: \qquad L_i(x_k) = \delta_{ik} := \begin{cases} 1, & k = i, \\ 0, & k \neq i. \end{cases}$$

Theorem 6.1 (interpolation polynomial)

For distinct nodes $x_0, \ldots, x_n$ and arbitrary values $y_0, \ldots, y_n$ there is exactly one $p \in \mathcal{P}_n$ with $p(x_i) = y_i$ for $i = 0, \ldots, n$. It is given by the **Lagrange form**

$$p(x) = \sum_{i=0}^{n} y_i \, L_i(x).$$

Blackboard: proof.

Example by hand

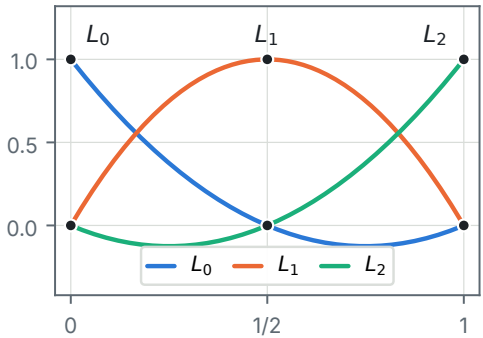
Nodes $0, \frac{1}{2}, 1$, values $0, 1, 0$ (the values of $\sin(\pi x)$).

Compute p.

$y_0 = y_2 = 0$: only L_1 contributes,

$$p(x) = L_1(x) = \frac{(x-0)(x-1)}{(\frac{1}{2}-0)(\frac{1}{2}-1)} = 4x(1-x).$$

Each L_i is 1 at its own node and 0 at the other nodes.



How well does p approximate f *between* the nodes? $f - p$ vanishes at $x_0, \dots, x_n$ — just like the **node polynomial** $\omega(x) := \prod_{j=0}^{n} (x - x_j)$.

Theorem 6.2 (interpolation error)

Let $f \in C^{n+1}([a,b])$, let $x_0, \dots, x_n \in [a,b]$ be distinct nodes, and let $p \in \mathcal{P}_n$ be the interpolation polynomial of f at these nodes. Then for every $x \in [a,b]$ there is $\xi = \xi(x) \in (a,b)$ with

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\omega(x).$$

Consequently, $\displaystyle \max_{x\in[a,b]} |f(x) - p(x)| \le \frac{\max_{[a,b]} |f^{(n+1)}|}{(n+1)!} \max_{x\in[a,b]} |\omega(x)|.$

Blackboard: proof.

Two factors: $\max |f^{(n+1)}|/(n+1)!$ depends only on f; $\max |\omega|$ depends only on the **nodes**.

$n = 1$, $h := x_1 - x_0$: **How large can** $|\omega(x)| = (x - x_0)(x_1 - x)$ **be on** $[x_0, x_1]$**?**
At most $h^2/4$: two nonnegative factors with sum h (maximum at the midpoint). Hence

$$\max_{[x_0,x_1]} |f - p| \le \frac{h^2}{8} \max_{[x_0,x_1]} |f''|. \tag{6.1}$$

Lemma 6.3 (node polynomial for equidistant nodes)

Let $n \ge 1$ and $x_j = x_0 + jh$, $j = 0, \dots, n$, with $h > 0$. Then $\max_{x \in [x_0, x_n]} |\omega(x)| \le \frac{n!}{4} h^{n+1}$.

Example 6.4: interpolation of $\sin(\pi x)$

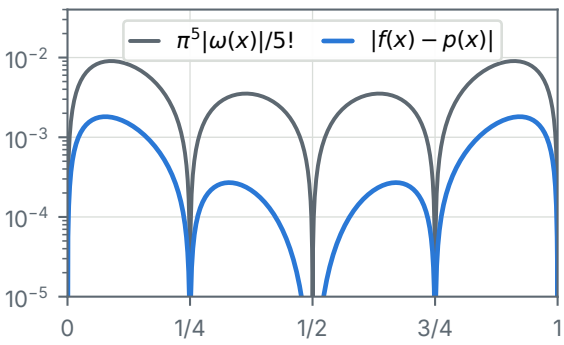
$f(x) = \sin(\pi x)$ on $[0,1]$, nodes $x_j = j/n$ ($n \geq 1$): $|f^{(n+1)}| \leq \pi^{n+1}$ and $h = \frac{1}{n}$, so Theorem 6.2 and Lemma 6.3 give

$$\max_{[0,1]} |f - p| \leq \frac{\pi^{n+1}}{(n+1)!} \cdot \frac{n!}{4n^{n+1}} = \frac{\pi^{n+1}}{4(n+1)\, n^{n+1}} \xrightarrow{n\to\infty} 0.$$

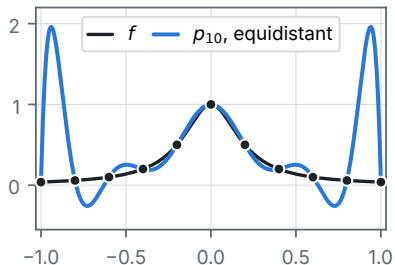
$n = 4$: bound ≈ 0.015. **Where is the error largest?**

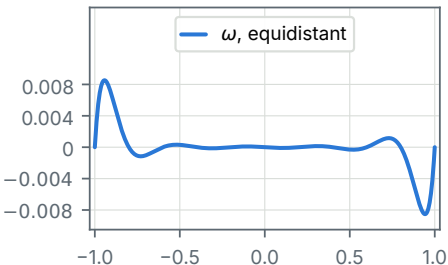
In the first and the last subinterval, where $|\omega|$ is largest. Actual maximum: $1.8 \cdot 10^{-3}$.

Convergence whenever $\max_{[a,b]} |f^{(k)}| \leq CR^k$ for all k (sin, cos, exp). **Not the general picture.**



Example 6.5 (Runge, 1901): $f(x) = \dfrac{1}{1+25x^2}$ on $[-1,1]$, equidistant nodes $x_j = -1 + \frac{2j}{n}$. f is infinitely differentiable. **What happens as n grows?**





$\max|f - p_n| = 1.9$ $(n=10)$, 60 $(n=20)$, $\to \infty$. $p_n(x) \to f(x)$ for $|x| < 0.7266\ldots$, $(p_n(x))$ unbounded for $0.7266\ldots < |x| < 1$.

No contradiction to Theorem 6.2: $|f^{(m)}(0)| = m!\,5^m$ for even m, so the bound does not tend to zero. The large errors sit where $|\omega|$ is large.

Idea: choose the nodes such that $\max|\omega|$ is as small as possible.

$\cos(2\theta) = 2\cos^2\theta - 1$. **Hence** $\cos(2\arccos x) =$ **?** $2x^2 - 1$ — a polynomial in x.

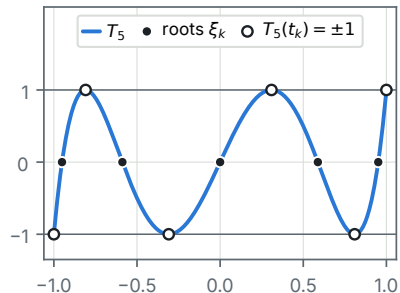
Lemma 6.6 (Chebyshev polynomials)

For $m \in \mathbb{N}_0$ let $T_m(x) := \cos(m \arccos x)$, $x \in [-1, 1]$.

(i) $T_0 = 1$, $T_1 = x$, $T_{m+1} = 2xT_m - T_{m-1}$ $(m \geq 1)$: a polynomial of degree m, with leading coefficient 2^{m-1} for $m \geq 1$.

(ii) $|T_m| \leq 1$ on $[-1, 1]$, and $T_m(t_k) = (-1)^k$ at $t_k := \cos\frac{k\pi}{m}$, $k = 0, \ldots, m$ $(m \geq 1)$.

(iii) For $m \geq 1$, T_m has the roots $\xi_k := \cos\frac{(2k+1)\pi}{2m}$, $k = 0, \ldots, m-1$; $T_m = 2^{m-1}\prod_k (x - \xi_k)$.

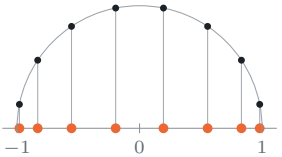


Theorem 6.7

Let $n \geq 0$. Among all monic polynomials of degree $n+1$, $2^{-n}T_{n+1}$ has the smallest maximum norm on $[-1,1]$, namely 2^{-n}. Consequently, for the **Chebyshev nodes**

$$x_k = \cos\left(\frac{2k+1}{2n+2}\,\pi\right), \qquad k = 0, \ldots, n$$

(the roots of T_{n+1}) we have $\omega = 2^{-n}T_{n+1}$, and $\max_{[-1,1]} |\omega| = 2^{-n}$ is minimal over all choices of $n+1$ nodes in $[-1,1]$.

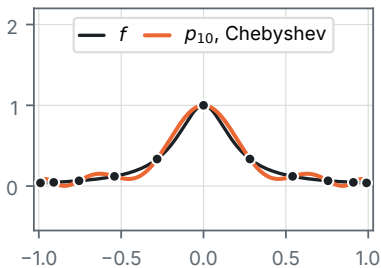


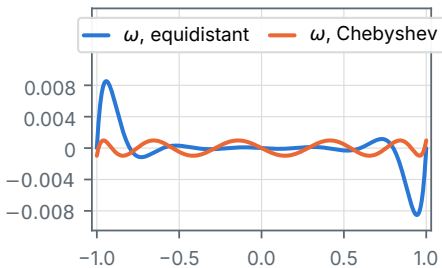
$n = 7$: equally spaced angles $\theta_k = \frac{2k+1}{16}\pi$, nodes $x_k = \cos\theta_k$

With Theorem 6.2: $\max\limits_{[-1,1]} |f - p| \leq \dfrac{\max_{[-1,1]} |f^{(n+1)}|}{2^n\,(n+1)!}$ — no other choice of $n+1$ nodes in $[-1,1]$ gives a smaller bound.

Chebyshev nodes for Runge's function

$n = 10$, Runge's function at the Chebyshev nodes. **What do you expect?**





Maximal error 0.11 ($n = 10$), 0.015 ($n = 20$) — equidistant: 1.9, 60.
$\max|\omega| = 2^{-10} \approx 0.00098$ versus 0.0085.

Only equidistant data? Keep the degree low, use many pieces: piecewise linear interpolation has the error bound $\frac{h^2}{8}\max|f''|$ by (6.1) $\rightarrow$ composite rules (Chapter 7).

Summary

Sections 5.6, 6.1–6.4

- **PageRank:** power method for $G = (1-d)A + \frac{d}{N}\mathbf{1}\mathbf{1}^\top$; error $\leq 2(1-d)^k$, independent of N.
- **Interpolation polynomial:** exists and is unique; Lagrange form $p = \sum_i y_i L_i$ with $L_i(x_k) = \delta_{ik}$.
- **Error:** $f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\omega(x)$ — one factor from f, one from the nodes.
- **Runge:** high degree at equidistant nodes can diverge, even for smooth f. **Chebyshev nodes** minimise $\max|\omega|$. Equidistant data: low degree, many pieces.
- **Tuesday:** Chapter 7 (numerical integration). Sheet 4 today; test 1 tomorrow, Friday 13 November, 13:45–14:15, HS I (Chapters 1–4).