Stochastic Processes
21. Skorohod's Theorem

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Skorohod's embedding theorem

- Theorem 16.28: $Y_1, Y_2, \ldots$ iid with $\mathrm{E}[Y_1] = 0$, and $S_n = Y_1 + \cdots + Y_n$. Then there is $(\Omega, \mathcal{F}, \mathrm{P})$ and $(\mathcal{F}_t)_{t \geq 0}$ and a Brownian motion $\mathcal{X} = (X_t)_{t \geq 0}$, which is a $(\mathcal{F}_t)_{t \geq 0}$ martingale and stopping times $T_1, T_2, \ldots$, so that:
  1. $(X_{T_1}, X_{T_2}, \ldots) \sim S_1, S_2, \ldots$ and
  2. $(T_{n+1} - T_n)_{n=0,1,2,\ldots}$ are independent with $\mathrm{E}[T_{n+1} - T_n] = \mathrm{V}[Y_1]$ for $n = 1, 2, \ldots$

Basic Lemma 1

- Lemma 16.23: Let Y have $\mathrm{E}[Y]=0$. For $w<0<z$ let $Y_{w,z}$ with states $\{w,z\}$ and $\mathrm{P}(Y_{w,z}=w)=\frac{z}{z+|w|}$. Then there is (W,Z) with $W\le 0, Z\ge 0$, so that $Y\sim Y_{W,Z}$.
- Proof: Let $Y\sim\mu$ (wlog $Y\neq 0$ as) and set $c=\mathrm{E}[Y^+]=\mathrm{E}[Y^-]$, as well as $(W,Z)\sim\mu_{W,Z}$ with

$$\mu_{W,Z}(dw,dz)=\frac{1}{c}(z+|w|)1_{w\le 0}1_{z\ge 0}\mu(dw)\mu(dz).$$

For $f:\mathbb{R}\to\mathbb{R}_+$ measurable with $f(0)=0$,

$$\begin{aligned}c\cdot\mathrm{E}[f(Y)]&=\mathrm{E}[Y^+]\cdot\mathrm{E}[f(-Y^-)]+\mathrm{E}[Y^-]\cdot\mathrm{E}[f(Y^+)]\\&=\int\int(zf(w)+|w|f(z))1_{z\ge 0}1_{w\le 0}\mu(dw)\mu(dz)\\&=\int\int(z+|w|)\mathrm{E}[f(Y_{w,z})]1_{z\ge 0}1_{w\le 0}\mu(dw)\mu(dz).\end{aligned}$$

Basic Lemma 2

- Lemma 16.25: Let Y habe $\mathsf{E}[Y] = 0$ and (W, Z) as above, $\mathcal{X} = (X_t)_{t \geq 0}$ an independent Brownian motion. Then

$$T_{W,Z} = \inf\{t \geq 0 : X_t \in \{W, Z\}\} \text{ is a stopping time},$$

$$X_{T_{W,Z}} \sim Y, \qquad \mathsf{E}[T_{W,Z}] = \mathsf{E}[Y^2].$$

- Proof: $X_{T_{w,z}}$ has values in $\{w, z\}$ and $(X_{T_{w,z} \wedge t})_{t \geq 0}$ is a martingale which, converges in L^1 against $X_{T_{w,z}}$. Therefore,

$$0 = \mathsf{E}[X_{T_{w,z}}] = w\mathsf{P}(X_{T_{w,z}} = w) + z(1 - \mathsf{P}(X_{T_{w,z}} = w)), \text{ or}$$
$$\mathsf{P}(X_{T_{w,z}} = w) = \frac{z}{z + |w|}$$

So $X_{T_{w,z}} \sim Y_{w,z} \sim Y$. By the quadratic variation of $\mathcal{X}$,

$$\mathsf{E}[T_{W,Z}] = \mathsf{E}[\mathsf{E}[T_{W,Z}|W, Z]] = \mathsf{E}[\lim_{t \to \infty} \mathsf{E}[T_{W,Z} \wedge t]|W, Z]$$
$$= \mathsf{E}[\mathsf{E}[X^2_{T_{W,Z}}|W, Z]] = \mathsf{E}[X^2_{T_{W,Z}}] = \mathsf{E}[Y^2].$$

Skorohod's embedding theorem

- Theorem 16.28: $Y_1, Y_2, \dots$ iid with $E[Y_1] = 0$, and $S_n = Y_1 + \cdots + Y_n$. Then there is $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t \geq 0}$ and a Brownian motion $\mathcal{X} = (X_t)_{t \geq 0}$, which is a $(\mathcal{F}_t)_{t \geq 0}$ martingale and stopping times $T_1, T_2, \dots$, so that:
  1. $(X_{T_1}, X_{T_2}, \dots) \sim S_1, S_2, \dots$ and
  2. $(T_{n+1} - T_n)_{n=0,1,2,\dots}$ are independent with $E[T_{n+1} - T_n] = V[Y_1]$ for $n = 1, 2, \dots$
- Proof: $\mathcal{X}$ is an independent BM. Let $(W_1, Z_1), (W_2, Z_2), \dots$ and $0 \leq T_1 < T_2 < \cdots$ be the stopping times of increments $(W_1, Z_1), (W_2, Z_2), \dots$ It follows from above that

$$(X_{T_1}, X_{T_2} - X_{T_1}, \dots) \sim (Y_1, Y_2, \dots),$$

$$(X_{T_1}, X_{T_2}, \dots) \sim (S_1, S_2, \dots),$$

$$E[T_{n+1} - T_n] = V[Y_1].$$

Random walk $\Rightarrow$ BM

- Corollary 16.28: $Y_1, Y_2, \dots$ iid with $E[Y_1] = 0$, $V[Y_1] = 1$, and $S_n = Y_1 + \cdots + Y_n$. Then there is a Brownian motion $\mathcal{X}$ with
$$\sup_{0 \le s \le t} \left| \frac{1}{\sqrt{n}} S_{[sn]} - \frac{1}{\sqrt{n}} X_{sn} \right| \xrightarrow{n \to \infty}_p 0, \qquad t > 0.$$
- Theorem 16.29: For the random walk above,
$$\limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1$$
almost surely.