

The background of the slide features a large, light blue watermark of the University of Vienna seal. The seal is circular and contains a central figure, likely a scholar or saint, seated and holding a book. Above the figure are three smaller figures in niches. The seal is surrounded by Latin text. The text 'Stochastics' is visible on the left side, and 'University of Vienna' is visible on the right side.

# Stochastic Processes

## 21. Skorohod's Theorem

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## Skorohod's embedding theorem

- Theorem 16.28:  $Y_1, Y_2, \dots$  iid with  $E[Y_1] = 0$ , and  $S_n = Y_1 + \dots + Y_n$ . Then there is  $(\Omega, \mathcal{F}, P)$  and  $(\mathcal{F}_t)_{t \geq 0}$  and a Brownian motion  $\mathcal{X} = (X_t)_{t \geq 0}$ , which is a  $(\mathcal{F}_t)_{t \geq 0}$  martingale and stopping times  $T_1, T_2, \dots$ , so that:
1.  $(X_{T_1}, X_{T_2}, \dots) \sim S_1, S_2, \dots$  and
  2.  $(T_{n+1} - T_n)_{n=0,1,2,\dots}$  are independent with  $E[T_{n+1} - T_n] = V[Y_1]$  for  $n = 1, 2, \dots$

## Basic Lemma 1

- ▶ Lemma 16.23: Let  $Y$  have  $E[Y] = 0$ . For  $w < 0 < z$  let  $Y_{w,z}$  with states  $\{w, z\}$  and  $P(Y_{w,z} = w) = \frac{z}{z+|w|}$ . Then there is  $(W, Z)$  with  $W \leq 0, Z \geq 0$ , so that  $Y \sim Y_{W,Z}$ .
- ▶ Proof: Let  $Y \sim \mu$  (wlog  $Y \neq 0$  as) and set  $c = E[Y^+] = E[Y^-]$ , as well as  $(W, Z) \sim \mu_{W,Z}$  with

$$\mu_{W,Z}(dw, dz) = \frac{1}{c}(z + |w|)1_{w \leq 0}1_{z \geq 0}\mu(dw)\mu(dz).$$

For  $f : \mathbb{R} \rightarrow \mathbb{R}_+$  measurable with  $f(0) = 0$ ,

$$\begin{aligned}c \cdot E[f(Y)] &= E[Y^+] \cdot E[f(-Y^-)] + E[Y^-] \cdot E[f(Y^+)] \\&= \int \int (zf(w) + |w|f(z))1_{z \geq 0}1_{w \leq 0}\mu(dw)\mu(dz) \\&= \int \int (z + |w|)E[f(Y_{w,z})]1_{z \geq 0}1_{w \leq 0}\mu(dw)\mu(dz).\end{aligned}$$

## Basic Lemma 2

- ▶ Lemma 16.25: Let  $Y$  have  $E[Y] = 0$  and  $(W, Z)$  as above,  $\mathcal{X} = (X_t)_{t \geq 0}$  an independent Brownian motion. Then

$T_{W,Z} = \inf\{t \geq 0 : X_t \in \{W, Z\}\}$  is a stopping time,

$$X_{T_{W,Z}} \sim Y, \quad E[T_{W,Z}] = E[Y^2].$$

- ▶ Proof:  $X_{T_{w,z}}$  has values in  $\{w, z\}$  and  $(X_{T_{w,z} \wedge t})_{t \geq 0}$  is a martingale which, converges in  $L^1$  against  $X_{T_{w,z}}$ . Therefore,

$$0 = E[X_{T_{w,z}}] = wP(X_{T_{w,z}} = w) + z(1 - P(X_{T_{w,z}} = w)), \text{ or}$$
$$P(X_{T_{w,z}} = w) = \frac{z}{z + |w|}$$

So  $X_{T_{w,z}} \sim Y_{w,z} \sim Y$ . By the quadratic variation of  $\mathcal{X}$ ,

$$E[T_{W,Z}] = E[E[T_{W,Z} | W, Z]] = E[\lim_{t \rightarrow \infty} E[T_{W,Z} \wedge t | W, Z]]$$
$$= E[E[X_{T_{W,Z}}^2 | W, Z]] = E[X_{T_{W,Z}}^2] = E[Y^2].$$

## Skorohod's embedding theorem

- ▶ Theorem 16.28:  $Y_1, Y_2, \dots$  iid with  $E[Y_1] = 0$ , and  $S_n = Y_1 + \dots + Y_n$ . Then there is  $(\Omega, \mathcal{F}, P)$  and  $(\mathcal{F}_t)_{t \geq 0}$  and a Brownian motion  $\mathcal{X} = (X_t)_{t \geq 0}$ , which is a  $(\mathcal{F}_t)_{t \geq 0}$  martingale and stopping times  $T_1, T_2, \dots$ , so that:
  1.  $(X_{T_1}, X_{T_2}, \dots) \sim S_1, S_2, \dots$  and
  2.  $(T_{n+1} - T_n)_{n=0,1,2,\dots}$  are independent with  $E[T_{n+1} - T_n] = V[Y_1]$  for  $n = 1, 2, \dots$
- ▶ Proof:  $\mathcal{X}$  is an independent BM. Let  $(W_1, Z_1), (W_2, Z_2), \dots$  and  $0 \leq T_1 < T_2 < \dots$  be the stopping times of increments  $(W_1, Z_1), (W_2, Z_2), \dots$ . It follows from above that

$$(X_{T_1}, X_{T_2} - X_{T_1}, \dots) \sim (Y_1, Y_2, \dots),$$

$$(X_{T_1}, X_{T_2}, \dots) \sim (S_1, S_2, \dots),$$

$$E[T_{n+1} - T_n] = V[Y_1].$$

## Random walk $\Rightarrow$ BM

- ▶ Corollary 16.28:  $Y_1, Y_2, \dots$  iid with  $E[Y_1] = 0$ ,  $V[Y_1] = 1$ , and  $S_n = Y_1 + \dots + Y_n$ . Then there is a Brownian motion  $\mathcal{X}$  with

$$\sup_{0 \leq s \leq t} \left| \frac{1}{\sqrt{n}} S_{[sn]} - \frac{1}{\sqrt{n}} X_{sn} \right| \xrightarrow[n \rightarrow \infty]{p} 0, \quad t > 0.$$

- ▶ Theorem 16.29: For the random walk above,

$$\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1$$

almost surely.