Stochastic Processes
7. The discrete stochastic integral

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Doob decomposition

- Definition 14.8: $\mathcal{X}$ is called $(\mathcal{F}_t)_{t\in I}$-previsible if $X_0 = 0$ and X_t is $\mathcal{F}_{t-1}$-measurable, $t = 1, 2, ...$
- Proposition 14.9: $\mathcal{X} = (X_t)_{t\in I}$ is adapted. Then,

  $\mathcal{X} = \mathcal{M} + \mathcal{A}$, where $\mathcal{M}$ is a martingale and $\mathcal{A}$ is previsible.

  $\mathcal{X}$ submartingale $\iff$ $\mathcal{A}$ almost surely non-decreasing.
- Proof: Define $\mathcal{A} = (A_t)_{t\in I}$ predictable by

$$A_t = \sum_{s=1}^{t} \mathbf{E}[X_s - X_{s-1}|\mathcal{F}_{s-1}].$$

  Then, $\mathcal{M} = \mathcal{X} - \mathcal{A}$ is a martingale.

  Uniqueness: With $\mathcal{X} = \mathcal{M} + \mathcal{A}$,

  $A_t - A_{t-1} = \mathbf{E}[X_t - X_{t-1}|\mathcal{F}_{t-1}]$ for all $t = 1, 2, ...$

Quadratic Variation

- Definition 14.10: $\mathcal{X} = (X_t)_{t\in I}$ square integrable martingale. The predictable process $(\langle \mathcal{X} \rangle_t)_{t\in I}$, for which $(X_t^2 - \langle \mathcal{X} \rangle_t)_{t\in I}$ is a martingale, is called the *quadratic variation process* of $\mathcal{X}$.
- Proposition 14.11: If $\mathcal{X} = (X_t)_{t\in I}$ is a square integrable martingale, then

  $$\langle \mathcal{X} \rangle_t = \sum_{s=1}^{t} \mathbf{E}[X_s^2 - X_{s-1}^2 | \mathcal{F}_{s-1}] = \sum_{s=1}^{t} \mathbf{E}[(X_s - X_{s-1})^2 | \mathcal{F}_{s-1}]$$

  and

  $$\mathbf{E}[\langle X \rangle_t] = \sum_{s=1}^{t} \mathbf{E}[X_s^2 - X_{s-1}^2] = \mathbf{E}[X_t^2 - X_0^2] = \mathbf{V}[X_t - X_0].$$

Examples

- Let $S_t = \sum_{i=1}^t X_i$ be a martingale with $X_1, X_2, \ldots$ quadratically integrable. Then,

$$\langle \mathcal{S} \rangle_t = \sum_{s=1}^t \mathbf{E}[X_s^2].$$

- Let $S_t = \prod_{s=1}^t X_s$ be a martingale with $X_1, X_2, \ldots$ quadratically integrable. Then

$$\langle \mathcal{S} \rangle_t = \sum_{s=1}^t \mathbf{E}[(S_s - S_{s-1})^2 | \mathcal{F}_{s-1}] = \sum_{s=1}^t S_{s-1}^2 \mathbf{E}[(X_s - 1)^2 | \mathcal{F}_{s-1}]$$

$$= \sum_{s=1}^t S_{s-1}^2 \mathbf{V}[X_s].$$

Stochastic integral

- Definition 14.13: $\mathcal{H} = (H_t)_{t\in I}$ previsible and $\mathcal{X} = (X_t)_{t\in I}$ adapted. Define the stochastic integral $\mathcal{H} \cdot \mathcal{X} = ((\mathcal{H} \cdot \mathcal{X})_t)_{t\in I}$ by

$$(\mathcal{H} \cdot \mathcal{X})_t = \sum_{s=1}^{t} H_s(X_s - X_{s-1})$$

- If $\mathcal{X}$ is a martingale, then so is $\mathcal{H} \cdot \mathcal{X}$.

  Indeed,

$$\begin{aligned} \mathbf{E}[(\mathcal{H} \cdot \mathcal{X})_{t+1} - (\mathcal{H} \cdot \mathcal{X})_t | \mathcal{F}_t] &= \mathbf{E}[H_{t+1}(X_{t+1} - X_t) | \mathcal{F}_t] \\ &= H_{t+1}\mathbf{E}[X_{t+1} - X_t | \mathcal{F}_t] \\ &= 0. \end{aligned}$$

Characterization of martingales

- Proposition 14.14: $\mathcal{X} = (X_t)_{t \in I}$ adapted with $\mathbf{E}[|X_0|] < \infty$.
  1. $\mathcal{X}$ martingale $\iff$ $\mathcal{H} \cdot \mathcal{X}$ is a martingale for every $\mathcal{H} = (H_t)_{t \in I}$ predictable;
  2. $\mathcal{X}$ submartingale (supermartingale) $\iff$ $\mathcal{H} \cdot \mathcal{X}$ is a sub-martingale (super-martingale) for every $\mathcal{H} = (H_t)_{t \in I}$ predictable, non-negative.
- 1. $\Rightarrow$ clear; $\Leftarrow$: For $t \in I$ let $H_s := 1_{\{s=t\}}$ be predictable. Since $(\mathcal{H} \cdot \mathcal{X})_{t-1} = 0$ holds, it follows that

$$0 = \mathbf{E}[(\mathcal{H} \cdot \mathcal{X})_t | \mathcal{F}_{t-1}] = \mathbf{E}[X_t - X_{t-1} | \mathcal{F}_{t-1}] = \mathbf{E}[X_t | \mathcal{F}_{t-1}] - X_{t-1}.$$

Quadratic variation of stochastic integrals

- Let $\mathcal{X} = (X_t)_{t \in I}$ be a martingale and $\mathcal{H} = (H_t)_{t \in I}$ previsible.

  Then

$$\begin{aligned} \langle \mathcal{H} \cdot \mathcal{X} \rangle_t &= \sum_{s=1}^{t} \mathbf{E}[((\mathcal{H} \cdot \mathcal{X})_s - (\mathcal{H} \cdot \mathcal{X})_{s-1})^2 | \mathcal{F}_{s-1}] \\ &= \sum_{s=1}^{t} \mathbf{E}[H_s^2 (X_s - X_{s-1})^2 | \mathcal{F}_{s-1}] \\ &= \sum_{s=1}^{t} H_s^2 \cdot \mathbf{E}[(X_s - X_{s-1})^2 | \mathcal{F}_{s-1}], \end{aligned}$$

  in particular

$$\mathbf{V}[(\mathcal{H} \cdot \mathcal{X})_t] = \sum_{s=1}^{t} \mathbf{E}[H_s^2 \cdot (X_s - X_{s-1})^2].$$

Stochastic integral and betting

- Given, we bet on the behavior of a stochastic process $\mathcal{X} = (X_t)_{t=0,1,2,\dots}$. If you know $X_0, \dots, X_{t-1}$, you bet H_t on rising prices. Then

$$(\mathcal{H} \cdot \mathcal{X})_t = \sum_{s=1}^{t} H_s (X_s - X_{s-1})$$

is the profit realized up to t.

- If $\mathcal{X}$ is a martingale, then so is $\mathcal{H} \cdot \mathcal{X}$.

Petersburger Paradox

- $X_1, X_2, \ldots$ iid with $\mathbf{P}(X_1 = \pm 1) = \frac{1}{2}$ and $S_t = \sum_{s=1}^t X_s$, as well as

$$H_t := 2^{t-1} 1_{\{S_{t-1}=-(t-1)\}}.$$

Then

$$(\mathcal{H} \cdot \mathcal{S})_t = \sum_{i=1}^t H_i(S_i - S_{i-1}) = \sum_{i=1}^t H_i X_i \xrightarrow{t\to\infty} 1$$

However, for the total bet $\sum_{t=1}^\infty H_t$,

$$\mathbf{E}\Big[\sum_{t=1}^\infty H_t\Big] = \sum_{k=1}^\infty \frac{1}{2^k}(2^k - 1) = \infty.$$