Stochastic Processes
6. Martingales in discrete time

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November 2, 2024

Repetition

- Probability space $(\Omega, \mathcal{F}, \mathbf{P})$; set of times $I = \{0, 1, 2, ...\}$
- A family $\mathcal{X} = (X_t)_{t\in I}$ is called a a stochastic process.
- A family $(\mathcal{F}_t)_{t\in I}$ of σ-algebras with $\mathcal{F}_t \subseteq \mathcal{F}, t \in I$, is called an *filtration* if $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t$.
- $(X_t)_{t\in I}$ is said to be an $(\mathcal{F}_t)_{t\in I}$ *adapted* if X_t is measurable with respect to $\mathcal{F}_t$.
- $(\sigma(X_s : s \leq t))_{t\in I}$ is called the generated filtration.
- For $s \leq t$ and any $X \in \mathcal{L}^1(\mathbf{P})$,

$$\mathbf{E}[\mathbf{E}[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathbf{E}[X|\mathcal{F}_s].$$

A simple martingale

- Example 14.1: Let $X \in \mathcal{L}^1(\mathbb{P})$.

  Then $\mathcal{X} = (X_t)_{t \in I}$ with

  $$X_t = \mathbf{E}[X|\mathcal{F}_t]$$

  is a stochastic process. For $s \leq t$,

  $$\mathbf{E}[X_t|\mathcal{F}_s] = \mathbf{E}[\mathbf{E}[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathbf{E}[X|\mathcal{F}_s] = X_s.$$

Super-, Sub-Martingale

- Definition 14.2: Let $\mathcal{X} = (X_t)_{t \in I}$ be integrable. Then $\mathcal{X}$ is called

$$\text{a martingale if } \mathbf{E}[X_t|\mathcal{F}_s] = X_s \text{ for } s, t \in I,\ s < t,$$

$$\text{sub-martingale, if } \mathbf{E}[X_t|\mathcal{F}_s] \geq X_s \text{ for } s, t \in I,\ s < t,$$

$$\text{super martingaleif } \mathbf{E}[X_t|\mathcal{F}_s] \leq X_s \text{ for } s, t \in I,\ s < t.$$

- Equivalent, for discrete I,

$$\text{a martingale, if } \mathbf{E}[X_t|\mathcal{F}_{t-1}] = X_{t-1} \text{ for } t > 0,$$

$$\text{Sub-Martingale, if } \mathbf{E}[X_t|\mathcal{F}_{t-1}] \geq X_{t-1} \text{ for } t > 0,$$

$$\text{Super-Martingale, if } \mathbf{E}[X_t|\mathcal{F}_{t-1}] \leq X_{t-1} \text{ for } t > 0.$$

Example 14.4.1

- Let $X_1, X_2, \ldots$ be independent, integrable with $\mathbf{E}[X_i] = 0, i = 1, 2, \ldots$ and $\mathcal{F}_t := \sigma(X_1, \ldots, X_t)$, as well as

$$S_t := \sum_{i=1}^{t} X_i.$$

Then $(S_t)_{t \in I}$ is a martingale, because

$$\mathbf{E}[S_t | \mathcal{F}_{t-1}] = \mathbf{E}[S_{t-1} + X_t | \mathcal{F}_{t-1}] = S_{t-1} + \mathbf{E}[X_t | \mathcal{F}_{t-1}]$$

$$= S_{t-1} + \mathbf{E}[X_t] = S_{t-1},$$

If $\mathbf{E}[X_i] \geq 0, i = 1, 2, \ldots$, then $(S_t)_{t \geq 0}$ is a submartingale.

Example 14.4.2

- $X_1, X_2, \ldots$ independent, integrable with $\mathbf{E}[X_i] = 1, i = 1, 2, \ldots$

  and $\mathcal{F}_t := \sigma(X_1, \ldots, X_t)$, as well as

$$S_t := \prod_{i=1}^{t} X_i.$$

  Then, $(S_t)_{t \in I}$ is integrable, and a martingale, since

$$\mathbf{E}[S_t | \mathcal{F}_{t-1}] = \mathbf{E}[S_{t-1} X_t | \mathcal{F}_{t-1}] = S_{t-1} \cdot \mathbf{E}[X_t | \mathcal{F}_{t-1}]$$

$$= S_{t-1} \cdot \mathbf{E}[X_t] = S_{t-1},$$

  If $\mathbf{E}[X_i] \geq 1$, $i = 1, 2, \ldots$, then $(S_t)_{t \in I}$ is a submartingale.

Example 14.4.3

- $I = \{-1, -2, ...\}$ and $X_1, X_2, ...$ iid, integrierbar, sowie

$$S_t := \frac{1}{|t|}\sum_{i=1}^{|t|} X_i, \qquad t \in I$$

and $\mathcal{F}_t := \sigma(..., S_{t-1}, S_t)$. Then for $t \in I$

$$\begin{aligned}\mathbf{E}[S_t|\mathcal{F}_{t-1}] &= \mathbf{E}\Big[\frac{1}{|t|}\sum_{i=1}^{|t|} X_i \Big| S_{t-1}, S_{t-2}, ...\Big] = \frac{1}{|t|}\sum_{i=1}^{|t|}\mathbf{E}\Big[X_i\Big|\sum_{i=1}^{|t|+1} X_i\Big] \\ &= \mathbf{E}\Big[X_1\Big|\sum_{i=1}^{|t|+1} X_i\Big] = \frac{1}{|t-1|}\sum_{i=1}^{|t-1|} X_i = S_{t-1}.\end{aligned}$$

In particular,

$$\mathbf{E}[X_1|\mathcal{F}_t] = \mathbf{E}\Big[X_1\Big|\sum_{i=1}^{|t|} X_i\Big] = \frac{1}{|t|}\sum_{i=1}^{|t|} X_i = S_t.$$

Example 14.5: Branching Process

- $I = \{0, 1, 2, ...\}$, $X_i^{(t)}$) iid with values in $\{0, 1, 2, ...\}$, $\mu = \mathbf{E}[X_i^{(t)}]$. Set $Z_0 = k$ and
$$Z_{t+1} = \sum_{i=1}^{Z_t} X_i^{(t)}.$$
$\mathcal{Z} = (Z_t)_{t \in I}$ is a martingale $\iff$ $\mu = 1$. Indeed:
$$\mathbf{E}[Z_{t+1} - Z_t | \mathcal{F}_t] = \mathbf{E}\Big[\sum_{i=1}^{Z_t} X_i^{(t)} - Z_t | \mathcal{F}_t\Big] = (\mu - 1) Z_t.$$
$\mathcal{Z} = (Z_t)_{t \in I}$ submartingale (supermartingale) $\iff$ $\mu \geq 1$ ($\mu \leq 1$). Furthermore, $(Z_t / \mu^t)_{t=0,1,2,...}$ is a martingale because
$$\mathbf{E}[Z_{t+1} - \mu Z_t | \mathcal{F}_t] = \mu Z_t - \mu Z_t = 0.$$

Example 14.6: Martingales derived from Markov chains

- Let E at most countable and $P = (p_{ij})_{i,j\in E}$ a stochastic matrix, $f : E \to \mathbb{R}$ bounded. Define a Markov chain via

$$\mathbf{P}(X_t = y | X_{t-1} = x) = p_{xy}.$$

$$\mathbf{E}[f(X_{s+1}) - f(X_s) | \mathcal{F}_s] = \sum_{x\in E} p_{X_s,y}(f(y) - f(X_s)).$$

Set

$$M_t = f(X_t) - \sum_{s=1}^{t-1} \mathbf{E}[f(X_{s+1}) - f(X_s) | X_s].$$

Then

$$\mathbf{E}[M_t - M_{t-1} | \mathcal{F}_{t-1}] = \mathbf{E}[f(X_t) - f(X_{t-1}) | \mathcal{F}_{t-1}]$$
$$- \mathbf{E}[f(X_t) - f(X_{t-1}) | X_{t-1}] = 0.$$

Functions of martingales

- Proposition 14.7: $\mathcal{X} = (X_t)_{t \in I}$ stochastic process, $\varphi : \mathbb{R} \to \mathbb{R}$ convex, such that $\varphi(X) = (\varphi(X_t))_{t \in I}$ integrable. If

  1. $\mathcal{X}$ is a martingale or
  2. $\mathcal{X}$ is a submartingale and φ is non-decreasing,

  then $\varphi(\mathcal{X}) = (\varphi(X_t))_{t \in I}$ is a submartingale.

  Proof: For 1., by Jensen's inequality,

  $$\varphi(X_s) = \varphi(\mathbf{E}[X_t|\mathcal{F}_s]) \leq \mathbf{E}[\varphi(X_t)|\mathcal{F}_s].$$

  For 2.,

  $$\varphi(X_s) \leq \varphi(\mathbf{E}[X_t|\mathcal{F}_s]) \leq \mathbf{E}[\varphi(X_t)|\mathcal{F}_s].$$