Stochastic Processes
5. Progressive measurability

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Progressive measurability

- Definition 13.31: $\mathcal{X}$ is *progressively measurable*, if, for all $t \in I$,
$$\begin{cases} I \cap [0,t] \times \Omega & \to E \\ (s,\omega) & \mapsto X_s(\omega) \end{cases}$$
is $I \cap \mathcal{B}([0,t]) \otimes \mathcal{F}_s / \mathcal{B}(E)$-measurable.
- Lemma 13.32: If I is countable or $\mathcal{X}$ has right-continuous paths, $\mathcal{X}$ is progressively measurable.
- Proposition 13.33: Let $\mathcal{X} = (X_t)_{t \in I}$ be progressively measurable, and T a stopping time. Then $X_T : \omega \mapsto X_{T(\omega)}(\omega)$ is measurable with respect to $\{T < \infty\} \cap \mathcal{F}_T$.

Progressive measurability

- Lemma 13.32: If I is countable or $\mathcal{X}$ has right-continuous paths, $\mathcal{X}$ is progressively measurable.
- Let $t \in I$. We consider the mapping

$$Y : \begin{cases} I \cap [0,t] \times \Omega & \to E \\ (s,\omega) & \mapsto X_s(\omega). \end{cases}$$

I countable: $Y^{-1}(B) = \bigcup_{s\in I, s\leq t} \{s\} \times X_s^{-1}(B)$ measurable.

$\mathcal{X}$ has right-continuous paths: $X_s^n := X_{(2^{-n}\lceil 2^n s\rceil)\wedge t}$, $n = 1,2,\ldots$ and the corresponding Y_n. Then, $Y_n \xrightarrow{n\to\infty}_{as} Y$ and

$$Y_n^{-1}(B) = \bigcup_{k:(k+1)2^{-n}\leq t} [k2^{-n},(k+1)2^{-n}) \times X_{(k+1)2^{-n}}^{-1}(B)$$

measurable.

Progressive measurability

- Proposition 13.33: Let $\mathcal{X} = (X_t)_{t \in I}$ be progressively measurable, and T a stopping time. Then $X_T : \omega \mapsto X_{T(\omega)}(\omega)$ is measurable with respect to $\{T < \infty\} \cap \mathcal{F}_T$.
- Proof: To show: $\{X_T \in B, T \leq t\} \in \mathcal{F}_t$ for $B \in \mathcal{B}(E)$, $t \in I$.

  Wlog $T \leq t$.

  Write $X_T = Y_t \circ \psi$, where $\psi(\omega) := (T(\omega), \omega)$ is measurable with respect to $\mathcal{F}_t / (I \cap \mathcal{B}([0, t]) \otimes \mathcal{F}_t)$ and $Y_t(s, \omega) = X_s(\omega)$ according to condition $I \cap \mathcal{B}([0, t]) \otimes \mathcal{F}_t / \mathcal{B}(E)$-measurable.