

The background of the slide features a large, light blue watermark of the University of Bonn seal. The seal is circular and contains a central figure of a seated woman holding a book, surrounded by various heraldic symbols and Latin text.

Stochastic Processes

5. Progressive measurability

Peter Pfaffelhuber

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Progressive measurability

- ▶ Definition 13.31: \mathcal{X} is *progressively measurable*, if, for all $t \in I$,

$$\begin{cases} I \cap [0, t] \times \Omega & \rightarrow E \\ (s, \omega) & \mapsto X_s(\omega) \end{cases}$$

is $I \cap \mathcal{B}([0, t]) \otimes \mathcal{F}_s / \mathcal{B}(E)$ -measurable.

- ▶ Lemma 13.32: If I is countable or \mathcal{X} has right-continuous paths, \mathcal{X} is progressively measurable.
- ▶ Proposition 13.33: Let $\mathcal{X} = (X_t)_{t \in I}$ be progressively measurable, and T a stopping time. Then $X_T : \omega \mapsto X_{T(\omega)}(\omega)$ is measurable with respect to $\{T < \infty\} \cap \mathcal{F}_T$.

Progressive measurability

- ▶ Lemma 13.32: If I is countable or \mathcal{X} has right-continuous paths, \mathcal{X} is progressively measurable.
- ▶ Let $t \in I$. We consider the mapping

$$Y : \begin{cases} I \cap [0, t] \times \Omega & \rightarrow E \\ (s, \omega) & \mapsto X_s(\omega). \end{cases}$$

I countable: $Y^{-1}(B) = \bigcup_{s \in I, s \leq t} \{s\} \times X_s^{-1}(B)$ measurable.

\mathcal{X} has right-continuous paths: $X_s^n := X_{(2^{-n} \lceil 2^n s \rceil) \wedge t}$, $n = 1, 2, \dots$

and the corresponding Y_n . Then, $Y_n \xrightarrow{n \rightarrow \infty}_{as} Y$ and

$$Y_n^{-1}(B) = \bigcup_{k: (k+1)2^{-n} \leq t} [k2^{-n}, (k+1)2^{-n}) \times X_{(k+1)2^{-n}}^{-1}(B)$$

measurable.

Progressive measurability

- ▶ Proposition 13.33: Let $\mathcal{X} = (X_t)_{t \in I}$ be progressively measurable, and T a stopping time. Then $X_T : \omega \mapsto X_{T(\omega)}(\omega)$ is measurable with respect to $\{T < \infty\} \cap \mathcal{F}_T$.
- ▶ Proof: To show: $\{X_T \in B, T \leq t\} \in \mathcal{F}_t$ for $B \in \mathcal{B}(E)$, $t \in I$.

Wlog $T \leq t$.

Write $X_T = Y_t \circ \psi$, where $\psi(\omega) := (T(\omega), \omega)$ is measurable with respect to $\mathcal{F}_t / (I \cap \mathcal{B}([0, t]) \otimes \mathcal{F}_t)$ and $Y_t(s, \omega) = X_s(\omega)$ according to condition $I \cap \mathcal{B}([0, t]) \otimes \mathcal{F}_t / \mathcal{B}(E)$ -measurable.