Stochastic Processes
4. Filtrations and Stopping times

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Definition

$\mathcal{X} = (X_t)_{t \in I}$ sp defined on $(\Omega, \mathcal{F}, \mathbf{P})$.

- Definition 13.20 (Filtration):

  $(\mathcal{F}_t)_{t \in I}$ with $\mathcal{F}_t \subseteq \mathcal{F}, t \in I$, all σ-algebras is called *filtration* if

  $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t$.

  If $\mathcal{F}_t = \sigma(X_s : s \leq t)$, we say the filtration is *generated* by $\mathcal{X}$.

  $\mathcal{X}$ is *adapted* to $(\mathcal{F}_t)_{t \in I}$ if X_t is a $\mathcal{F}_t$-measurable, $t \in I$.

Definition

$\mathcal{X} = (X_t)_{t \in I}$ sp defined on $(\Omega, \mathcal{F}, \mathbf{P})$, $(\mathcal{F}_t)_{t \in I}$ a filtration.

- Definition 13.20 (Stopping time):

  A rv T with values in $\bar{I}$ (the completion of I) is called *random time*. It is a *stopping time* if $\{T \leq t\} \in \mathcal{F}_t$, $t \in I$. It is an *optional time* if $\{T < t\} \in \mathcal{F}_t$, $t \in I$.

  For a stopping time T, define

  $$\mathcal{F}_T := \{A \in \mathcal{A} : A \cap \{T \leq t\} \in \mathcal{F}_t, t \in I\}$$

  For a random time T, define $X_T : \omega \mapsto X_{T(\omega)}(\omega)$ and $\mathcal{X}^T := (X_{T \wedge t})_{t \in I}$, the process stopped at T.

Hitting times in the PPP

- Example 13.22: $\mathcal{X} = (X_t)_{t\in[0,\infty)}$ and $\mathcal{Y} = (Y_t)_{t\in[0,\infty)}$ the right and left continuous PPP, and $(\mathcal{F}_t^{\mathcal{X}})_{t\in[0,\infty)}$ and $(\mathcal{F}_t^{\mathcal{Y}})_{t\in 0,\infty)}$ the corresponding filtrations. Let

$$T_1 := \inf\{t \geq 0 : X_t = 1\} = \inf\{t \geq 0 : Y_t = 1\}.$$

Then: T_1 is both $(\mathcal{F}_t^{\mathcal{X}})_{t\in[0,\infty)}$-stopping time, as well as a $(\mathcal{F}_t^{\mathcal{X}})_{t\in[0,\infty)}$ option time.

- Proof: If $T_1 = t$ is the jump time from 0 to 1, then $X_t = 1$, i.e. $\{T_1 \leq t\} = \{X_t \geq 1\} \in \sigma((X_s)_{s\leq t}) = \mathcal{F}_t^{\mathcal{X}}$ and $\{T_1 < t\} = \{X_{t-} \geq 1\} \in \sigma((X_s)_{s<t}) \subseteq \mathcal{F}_t^{\mathcal{X}}$.

Propoerties of Stopping Times

From now on: Let $(\mathcal{F}_t)_{t\in I}$ be a filtration.

- Lemma 13.23:
  1. $T = s \in I$ is a stopping time, $s \in I$.
  2. S, T stopping times $\Rightarrow S \wedge T$ and $S \vee T$, $S + T$ are stopping times;
  3. T stopping time $\Rightarrow$ T is $\mathcal{F}_T$ measurable.
  4. S, T stopping times with $S \leq T \Rightarrow \mathcal{F}_S \subseteq \mathcal{F}_T$.
- Proof: Let $t \in I$. 1. Clear since $\{s \leq t\} \in \{\emptyset, \Omega\} \subseteq \mathcal{F}_t$.

  2. $\{S \wedge T \leq t\} = \{S \leq t\} \cup \{T \leq t\} \in \mathcal{F}_t$

  $\{S \vee T \leq t\} = \{S \leq t\} \cap \{T \leq t\} \in \mathcal{F}_t$.

  $S' + T'$ is $\mathcal{F}_t$-measurable for $S' = S \wedge t + 1_{\{S>t\}}$,

  $T' = T \wedge t + 1_{\{T>t\}}$ and $\{S + T \leq t\} = \{S' + T' \leq t\} \in \mathcal{F}_t$.

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  4. S, T stopping times with $S \leq T \Rightarrow \mathcal{F}_S \subseteq \mathcal{F}_T$.
- Proof: 4. T stopping time

  $\Rightarrow \{T \leq t\} \in \mathcal{F}_t \Rightarrow \{T \leq t\} \in \mathcal{F}_T$

  5. $A \in \mathcal{F}_S$ and $t > s$. Since $B := A \cap \{S \leq t\} \in \mathcal{F}_t$,

$$A \cap \{T \leq t\} = B \cap \{T \leq t\} \in \mathcal{F}_t \Rightarrow A \in \mathcal{F}_T$$

Complete and continuous filtrations

- Definition 13.24: Define $\mathcal{F}_t^+ := \bigcap_{s>t} \mathcal{F}_s$. We call $(\mathcal{F}_t)_{t\in[0,\infty)}$ *continuous* if $\mathcal{F}_t^+ = \mathcal{F}_t$.

  Let $\mathcal{N} = \{A : \exists N \supseteq A \text{ with } N \in \mathcal{F} \text{ and } \mathbf{P}(N) = 0\}$. Then, the filtration $(\mathcal{F}_t)_{t\in I}$ is *complete* if $\mathcal{N} \subseteq \mathcal{F}_t$ for each $t \in I$.
- Lemma 13.25: The filtration $(\mathcal{G}_t)_{t\in[0,\infty)}$ with

$$\mathcal{G}_t = \sigma(\mathcal{F}_t^+, \mathcal{N}).$$

  is the smallest continuous and complete filtration.

  Furthermore, $\mathcal{G}_t = \sigma(\mathcal{F}_t, \mathcal{N})^+$.

Option and stopping times

- Lemma 13.26: A random time T is an option time iff it is a $(\mathcal{F}_t^+)_{t\in[0,\infty)}$-stopping time. In this case,

$$\mathcal{F}_T^+ = \{A \in \mathcal{F} : A \cap \{T < t\} \in \mathcal{F}_t, t > 0\}.$$

  In particular, if $(\mathcal{F}_t)_{t\in[0,\infty)}$ is continuous, every stopping (option) time is an option (stopping) time.
- Proof: First, If T is a $(\mathcal{F}_t^+)_{t\in[0,\infty)}$-stopping time and $A \cap \{T \leq t\} \in \mathcal{F}_t^+$, Then,

$$A \cap \{T < t\} = \bigcup_{\mathbb{Q}\ni s<t} (A \cap \{T \leq s\}) \in \mathcal{F}_t.$$

  Set $A = \Omega$ for the first assertion, and general A for the second.

Suprema and infima of stopping times

- Lemma 13.27: $T_1, T_2, \dots$ random times. Then:
  1. $T_1, T_2, \dots$ stopping times $\Rightarrow$ $T := \sup_n T_n$ is a stopping time.
  2. $I = \{0, 1, 2, \dots\}$, $T_1, T_2, \dots$ stopping times $\Rightarrow$ $T := \inf_n T_n$ is a stopping time.
  3. $I = [0, \infty)$, $T_1, T_2, \dots$ option times $\Rightarrow$ $T := \inf_n T_n$ is an option time. In addition, $\mathcal{F}_T^+ = \bigcap_n \mathcal{F}_{T_n}^+$.
  4. Proof: 1. $\{\sup T_n \le t\} = \bigcap_n \{T_n \le t\} \in \mathcal{F}_t$

     2. $\{\inf T_n \le t\} = \bigcup_n \{T_n \le t\} \in \mathcal{F}_t$.

     3. $\{\inf T_n < t\} = \bigcup_n \{T_n < t\} \in \mathcal{F}_t$.

     Since $T \le T_n$, $\mathcal{F}_T^+ \subseteq \bigcap_n \mathcal{F}_{T_n}^+$. If $A \in \bigcap_n \mathcal{F}_{T_n}^+$, then

$$A \cap \{T < t\} = A \cap \bigcup_n \{T_n < t\} = \bigcup_n (A \cap \{T_n < t\}) \in \mathcal{F}_t.$$

     Thus $A \in \mathcal{F}_T^+$, so $\bigcap_n \mathcal{F}_{T_n}^+ \subseteq \mathcal{F}_T^+$.

Hitting times as option and stopping times

- Definition 13.29: The *hitting time of* $B \in \mathcal{B}(E)$ is given by

$$T_B := \inf\{t : X_t \in B\}.$$

- Proposition 13.30: $\mathcal{X} = (X_t)_{t \in I}$ adapted:
  1. If $I = \{0, 1, 2, ...\} \Rightarrow$, T_B is a stopping time.
  2. If $I = [0, \infty)$, B open, $\mathcal{X}$ has right-continuous paths $\Rightarrow$ T_B is option time.
  3. If $I = [0, \infty)$, B is closed, $\mathcal{X}$ has continuous paths $\Rightarrow$ T_B is stopping time.

Hitting times as option and stopping times

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  3. If $I = [0, \infty)$, B is closed, $\mathcal{X}$ has continuous paths $\Rightarrow$ T_B is stopping time.
- Proof: 1. $\{T_B \le t\} = \bigcup_{s\le t}\{X_s \in B\} \in \mathcal{F}_t$.

  2. $\{T_B < t\} = \bigcup_{\mathbb{Q}\ni s<t}\{X_s \in B\} \in \mathcal{F}_t$.

  3. with $B_n := \{x \in E : r(x, B) < 1/n\}$:

  $$\{T_B \le t\} = \bigcap_n \{T_{B_n} \le t\} = \bigcap_n (\{T_{B_n} < t\} \cup \{X_t \in \overline{B_n}\}) \in \mathcal{F}_t.$$

  This shows all assertions.