Stochastic Processes
3. Brownian Motion

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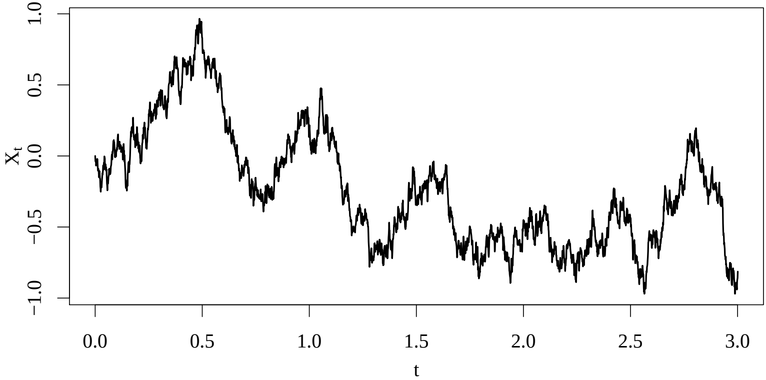
A path of a Brownian motion



Brownian Motion and Gaussian Processes

Definition 13.15:

- $\mathcal{X}$ is called *Gaussian* if $c_1 X_{t_1} + \cdots + c_n X_{t_n}$ is normal for all $c_1, \ldots, c_n \in \mathbb{R}$ and $t_1, \ldots, t_n \in I$.
- $t \mapsto \mathbf{E}[X_t]$ denotes its expectation and $(s,t) \mapsto \mathbf{COV}(X_s, X_t)$ its covariance structure.
- If $\mathcal{X}$ has continuous paths and $(X_{t_i} - X_{t_{i-1}})_{i=1,\ldots,n}$ is independent with $X_{t_i} - X_{t_{i-1}} \sim N(0, t_i - t_{i-1})$ for all $t_0 \leq \cdots \leq t_n$, $\mathcal{X}$ is a *Brownian motion* (BM).
- $\mathcal{X}^1, \mathcal{X}^d$ be independent BMs. Then, $(\mathcal{X}^i)_{i=1,\ldots,d}$ is a *d-dimensional Brownian motion*.

Existence of Brownian Motion

- Proposition 13.17: Let $\mathcal{X}$ be such that for $0 = t_0 < t_1 < ... < t_n$ it holds that $X_{t_i} - X_{t_{i-1}} \sim N(0, t_i - t_{i-1})$ are independent. Then, there exists a modification $\mathcal{Y}$ of $\mathcal{X}$ with continuous paths. It holds $\mathbf{COV}(X_s, X_t) = s \wedge t$.
- Proof: Existence, uniqueness as in Proposition 13.11. Since $X_s \sim N(0, s)$, $X_s \overset{d}{=} s^{1/2} X_1$. For $a > 2$,

  $\mathbf{E}[|X_t - X_s|^a] = \mathbf{E}[|X_{t-s}|^a] = \mathbf{E}[((t-s)^{1/2}|X_1|)^a] = (t-s)^{a/2}\mathbf{E}[|X_1|^a]$.

  With Theorem 13.8, $\mathcal{Y}$ exists. With $s \leq t$,

  $\mathbf{COV}(X_s, X_t) = \mathbf{COV}(X_s, X_s) + \mathbf{COV}(X_s, X_t - X_s) = \mathbf{V}[X_s] = s$.

Characterization of Gaussian Processes

- Lemma 13.18: Let $\mathcal{X} = (X_t)_{t\in[0,\infty)}$ and $\mathcal{Y} = (Y_t)_{t\in[0,\infty)}$ be Gaussian processes with the same expectation and covarience structure. Then, they are versions from each other.
- Proof: Since a normal distribution is uniquely determined by its expectation and covariance, the result follows from Proposition 13.6.1

Brownian Scaling

- Theorem 13.19: Let $\mathcal{X} = (X_t)_{t\in[0,\infty)}$ be a BM. Then, the processes $(X_{c^2t}/c)_{t\in[0,\infty)}$ are for each $c > 0$ and $(tX_{1/t})_{t\in[0,\infty)}$ also BM.
- Proof: By linearity, $(X_{c^2t}/c)_{t\in[0,\infty)}$ and $(tX_{1/t})_{t\in[0,\infty)}$ are Gaussian processes. Furthermore,

$$\mathbf{E}[X_{c^2t}/c] = 0, \qquad \mathbf{E}[tX_{1/t}] = 0,$$

and for $s, t \geq 0$

$$\mathbf{COV}[X_{c^2s}/c, X_{c^2t}/c] = \frac{1}{c^2}(c^2s \wedge c^2t) = s \wedge t,$$
$$\mathbf{COV}[sX_{1/s}, tX_{1/t}] = st\left(\frac{1}{s} \wedge \frac{1}{t}\right) = s \wedge t.$$

Now the assertion follows with the last lemma.