Stochastic Processes
2. The Poisson process

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The Poisson process

- Remark 3.9: We want to model a count process with the following propertiers:
    1. *Independent increments:* If $0 = t_0 < t_1 < ... < t_n$, then $(X_{t_i} - X_{t_{i-1}} : i = 1, ..., n)$ is an independent family.
    2. *Identically distributed increments:* If $0 < t_1 < t_2$, then $X_{t_2} - X_{t_1} \stackrel{d}{=} X_{t_2 - t_1} - X_0$.
    3. *No double-points:* $\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbf{P}(X_\varepsilon - X_0 > 1) = 0$.
- Definition 13.10: $\mathcal{X} = (X_t)_{t \in [0,\infty)}$ is a Poisson (point) process with intensity λ (PPP(λ)) iff:
    1. For $0 = t_0 < ... < t_n$, the family $(X_{t_i} - X_{t_{i-1}} : i = 1, ..., n)$ is independent.
    2. For $0 \leq t_1 < t_2$ is $X_{t_2} - X_{t_1} \sim \text{Poi}(\lambda(t_2 - t_1))$.

The Poisson process

- Proposition 13.11: Let $\lambda \geq 0$. There is exactly one PPP(λ).
- Proof: Uniqueness follows from uniqueness of fdds.
- Existence using a Projective Limit: Let

  $J = \{t_1 < ... < t_n\} \subseteq_f I$,

  $S^n(x_1 - x_0, ..., x_n - x_{n-1}) := (x_1, ..., x_n)$, and

  $$\mathbf{P}_J := S^n_* \bigotimes_{i=1}^n \mathrm{Poi}(\lambda(t_i - t_{i-1})).$$

  Then, $(\mathbf{P}_J : J \subseteq_f I)$ is projective since

  $$\mathrm{Poi}(\lambda(t_{i+1} - t_i)) * \mathrm{Poi}(\lambda(t_i - t_{i-1})) = \mathrm{Poi}(\lambda(t_{i+1} - t_{i-1})).$$

  Existence now follows with Throrem 5.24.

Characterization of Poisson processes

- Proposition 13.12: $\mathcal{X} = (X_t)_{t \in I}$ non-decreasing with $X_0 = 0$, values in $\mathbb{Z}_+$ is PPP(λ) iff $\lambda = \mathbf{E}[X_1 - X_0] < \infty$ and 1.-3. from Remark 13.9 hold.
- Proof: '$\Rightarrow$': 1. and 2. ✓. For 3.

$$\frac{1}{\varepsilon}\mathbf{P}(X_\varepsilon > 1) = \frac{1 - e^{-\lambda\varepsilon}(1+\lambda\varepsilon)}{\varepsilon} \leq \frac{1-(1-\lambda\varepsilon)(1+\lambda\varepsilon)}{\varepsilon} \xrightarrow{\varepsilon \to 0} 0.$$

Characterization of Poisson processes

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- Proof: '$\Leftarrow$': 1. ✓.

  To show: $X_t \sim \text{Poi}(\lambda t)$. Let for $n \in \mathbb{N}, k = 1, ..., n$,

  $$Z_k^n := (X_{tk/n} - X_{t(k-1)/n}) \wedge 1, \qquad X_t^n = \sum_{k=1}^n Z_k^n \sim B\big(n, \mathbf{P}(X_{t/n} > 0$$

  $$\begin{aligned}\mathbf{P}(\lim_{n\to\infty} X_t^n \neq X_t) &= \lim_{n\to\infty} \mathbf{P}(X_t^n \neq X_t) \\ &\leq \lim_{n\to\infty} \sum_{k=1}^n \mathbf{P}(X_{tk/n} - X_{t(k-1)/n} > 1) \\ &= \lim_{n\to\infty} n\mathbf{P}(X_{t/n} > 1) \xrightarrow{n\to\infty} 0\end{aligned}$$

Characterization of Poisson processes

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- Proof: '$\Leftarrow$': from 3. Since $X_t^n \uparrow X_t$, by monotone convergence,

$$\lambda t = \mathbf{E}[X_t] = \lim_{n\to\infty} \mathbf{E}[X_t^n] = \lim_{n\to\infty} np_n.$$

By a Poisson approximation,

$$\mathbf{P}(X_t = k) = \lim_{n\to\infty} \mathbf{P}(X_t^n = k) = \mathrm{Poi}(\lambda t)(k),$$

i.e. $X_t \sim \mathrm{Poi}(\lambda t)$ and the assertion follows.

Construction by exponential distributions

- Proposition 13.13: $S_1, S_2, ... \sim \exp(\lambda)$ be iid, $\mathcal{X} = (X_t)_{t \in I}$ given by
$$X_t := \max\{i : S_1 + ... + S_i < t\}$$
with $\max \emptyset = 0$. Then $\mathcal{X}$ is a PPP(λ).
- Proof, special case. We write, with $U_1, ..., U_k \sim U([0,1])$ iid
$$\begin{aligned}
&\mathbf{P}(X_t) \\
&= \int_0^t \int_{t_1}^t \cdots \int_{t_{k-1}}^t \lambda^k e^{-\lambda t_1} e^{-\lambda(t_2 - t_1)} \cdots e^{-\lambda(t_k - t_{k-1})} e^{-\lambda(t - t_k)} dt_k ... dt_1 \\
&= e^{-\lambda t} \lambda^k \frac{t^k}{t^k} \int_0^t \int_{t_1}^t \cdots \int_{t_{k-1}}^t dt_k \cdots dt_1 \\
&= e^{-\lambda t} \lambda^k t^k \mathbf{P}[U_1 < ... < U_k] = e^{-\lambda t} \lambda^k t^k \frac{1}{k!} = \mathrm{Poi}(\lambda t)(k).
\end{aligned}$$

The right- and left-continuous PPP

- In the setting above, set

$$X_t := \max\{i : S_1 + \dots + S_i < t\},$$

$$Y_t := \max\{i : S_1 + \dots + S_i \le t\}.$$

Then, $\mathcal{X}$ and $\mathcal{Y}$ are modifications.