Stochastic Processes
1. Definition

Peter Pfaffelhuber

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Stochastic process

$(\Omega, \mathcal{F}, \mathrm{P})$ probability space; I some index set, E some set.

- Definition 13.1.: $\mathcal{X} = (X_t)_{t \in I}$ such that $X_t : \Omega \to E$ is $\mathcal{F}/\mathcal{B}(E)$-measurable is called an E-valued *(stochastic) process*. For $\omega \in \Omega$ fixed, $X(\omega) : t \mapsto X_t(\omega)$ is called a *path* of X.
- If $\Omega = E^I$ and $X_t = \pi_t$, $\mathcal{X}$ is called *canonical process*.
- Let $0 < p < \infty$ and $E = \mathbb{R}$. $\mathcal{X}$ is p-times integrable if $\mathrm{E}[|X_t|^p] < \infty$ for all $t \in I$.

  It is L^p-bounded, if $\sup_{t \in I} \mathrm{E}[|X_t|^p] < \infty$.

Examples

- Let $(X_t)_{t\in I}$ be independent. Then $\mathcal{X} = (X_t)_{t\in I}$ is a (very simple) stochastic process (sp).
- Let $X_1, X_2, \dots$ real-valued, independent, identically distributed random variables. Then, $\mathcal{S} = (S_t)_{t=0,1,2,\dots}$ with $S_0 = 0$ and
$$S_t = \sum_{i=1}^{t} X_i$$
If $\mathrm{P}(X_i = \pm 1) = 1/2$, then $\mathcal{S}$ is a simple random walk.
- Let $\kappa(.,.)$ be a stochastic kernel (see Definition 5.9) from $(E, \mathcal{B}(E))$ to $(E, \mathcal{B}(E))$ and X_0 a random variable. Given X_t, X_{t+1} is $\kappa(X_t, .)$-distributed, $t = 0, 1, 2, \dots$ Then $(X_t)_{t=0,1,\dots}$ is called an E-valued *Markov chain*.

Existence of stochastic processes

- From Section 5: a projective family on $\mathcal{F}$ is a family of distributions $(\mathrm{P}_J)_{J \subseteq_f I}$ with $\mathrm{P}_H = (\pi_H^J)_* \mathrm{P}_J$ for $H \subseteq J$, where π_H^J is the projection of E^J onto E^H.
- If $\mathcal{X} = (X_t)_{t \in I}$ is given, $(X_{t_1}, ..., X_{t_n})$ is called a finite-dimensional distribution of $\mathcal{X}$. The fdd-family is projective; see also Example 5.22.2.
- For given fdd-distributions, the Kolmogorov's extension theorem guarantees existence of (the distribution of the) sp (if E is Polish).

Equality of spes

- Definition 13.4: $\mathcal{X} = (X_t)_{t\in I}$ and $\mathcal{Y} = (Y_t)_{t\in I}$ two spes.
  1. If $\mathcal{X} \stackrel{d}{=} \mathcal{Y}$, then $\mathcal{Y}$ is a *version* of $\mathcal{X}$.
  2. If $\mathrm{P}(X_t = Y_t) = 1$ for all $t \in I$, then $\mathcal{X}$ is a *modification* of $\mathcal{Y}$.
  3. If $\mathrm{P}(X_t = Y_t \text{ for all } t \in I) = 1$, then $\mathcal{X}$ and $\mathcal{Y}$ are called *indistinguishable*.
- Definition 13.5: $f : I \to E$ is rcll (or càdlàg) if, for all $t \in I$,

$$f(t) = \lim_{s\downarrow t} f(s) \text{ and } \lim_{s\uparrow t} f(s) \text{ exists.}$$

  The set of right-continuous functions with left limits is denoted by $\mathcal{D}_E(I)$.

Interplay of equality of spes

- Proposition 13.6: Let $\mathcal{X}$ and $\mathcal{Y}$ spes.
  1. $\mathcal{X} \overset{d}{=} \mathcal{Y}$ iff $(X_{t_1}, \dots, X_{t_n}) \overset{d}{=} (Y_{t_1}, \dots, Y_{t_n})$ for any choice of $n \in \mathbb{N}$ and $t_1, \dots, t_n \in I$.
  2. $\mathcal{X}$, $\mathcal{Y}$ indistinguishable $\Rightarrow$ $\mathcal{X}$ modification of $\mathcal{Y}$ $\mathcal{X}$ modification of $\mathcal{Y}$ $\Rightarrow$ $\mathcal{X}$ version of $\mathcal{Y}$.
  3. I at most countable, $\mathcal{X}$ modification of $\mathcal{Y}$ $\Rightarrow$ $\mathcal{X}$, $\mathcal{Y}$ indistinguishable.
  4. Let $I = [0, \infty)$, $\mathcal{X}$, $\mathcal{Y}$ have right-continuous paths, $\mathcal{X}$ modification of $\mathcal{Y}$ $\Rightarrow$ $\mathcal{X}$, $\mathcal{Y}$ indistinguishable.

Interplay of equality of spes

- 1. Consider the $\cap$-stable generator

$$\mathcal{C} := \{\pi_J^{-1}(A) : A \in \mathcal{B}(E)^{|J|}; J \subseteq_f I\} \subseteq \mathcal{B}(E)^I$$

of $\mathcal{B}(E)^I$. Since $\mathcal{X}_*\mathrm{P}$ and $\mathcal{Y}_*\mathrm{P}$ agree on $\mathcal{C}$, they are the same; see Theorem 2.11.
- 2. Let $t \in I$. So, $\mathrm{P}(X_t \neq Y_t) \leq \mathrm{P}(X_s \neq Y_s \text{ for a } s \in I) = 0$. If $\mathcal{X}$ and $\mathcal{Y}$ are modifications and $t_1, \ldots, t_n \in I$, then $\mathrm{P}(X_{t_1} = Y_{t_1}, \ldots, X_{t_n} = Y_{t_n}) = 1$ since finite unions of null-sets are null-sets. The rest follows by 1.
- 3.The statement is clear because

$$\mathrm{P}(X_t \neq Y_t \text{ for a } t \in I) \leq \sum_{t \in I} \mathrm{P}(X_t \neq Y_t) = 0.$$

Interplay of equality of spes

- 4. Let R be a set with $\mathrm{P}(R) = 1$ such that $\mathcal{X}$ and $\mathcal{Y}$ have right-continuous paths on R and $N_t := \{X_t \neq Y_t\}$. Further, let $I' = I \cap \mathbb{Q}$. Then, $\mathrm{P}(\bigcup_{t \in I'} N_t) = 0$ and

$$\mathrm{P}\Big(\bigcup_{t \in I} N_t\Big) \leq \mathrm{P}\Big(R \cap \bigcup_{t \in I} \bigcup_{r \geq t, r \in I'} N_r\Big) = \mathrm{P}\Big(R \cap \bigcup_{r \in I'} N_r\Big) = 0.$$

Versions and path properties

- Let $\mathcal{X}$ be a modification of $\mathcal{Y}$ and $\mathcal{Y}$ has continuous paths.

  Then, $\mathcal{X}$ does not need to have continuous paths.

  Indeed: Let $\mathcal{Y} = 0$, as well as $T \sim \exp(1)$ and $\mathcal{X} = (X_t)_{t \in I}$ given by

$$X_t = \begin{cases} 1, & t = T, \\ 0, & \text{otherwise.} \end{cases}$$

  Then $\mathrm{P}(X_t = Y_t) = \mathrm{P}(T \neq t) = 1$ for each $t \in I$. Note that every path of $\mathcal{X}$ is discontinuous (at T).

Theorem by Kolmogorov-Chentsov

- Theorem 13.8: Let $I = [0, \infty)$ and $\mathcal{X}$ be a sp (with values in (E, r)). For every $\tau > 0$ there are numbers $\alpha, \beta, C > 0$ with

  $$\mathrm{E}[r(X_s, X_t)^\alpha] \leq C|t - s|^{1+\beta}, \qquad 0 \leq s, t \leq \tau.$$

  Then there is a modification $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ with continuous paths.

  The paths are even almost surely local Hölder-continuous of any order $\gamma \in (0, \beta/\alpha)$.
- Proof: Wlog $I = [0, 1]$. Set

  $$D_n := \{0, 1, ..., 2^n\} \cdot 2^{-n}, \qquad n = 0, 1, ...$$

  and $D = \bigcup_{n=0}^\infty D_n$ and

  $$\xi_n := \max\{r(X_s, X_t) : s, t \in D_n, |t - s| = 2^{-n}\}.$$

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- Proof: Let $0 < \gamma < \beta/\alpha$. Then for some $C > 0$,

$$\begin{aligned}
\mathrm{E}\Big[\sum_{n=0}^{\infty}(2^{\gamma n}\xi_n)^\alpha\Big] &= \sum_{n=0}^{\infty} 2^{\alpha\gamma n}\mathrm{E}[\xi_n^\alpha] \\
&\leq \sum_{n=0}^{\infty} 2^{\alpha\gamma n} \sum_{s,t\in D_n, |t-s|=2^{-n}} \mathrm{E}[r(X_s, X_t)^\alpha] \\
&\leq C\sum_{n=0}^{\infty} 2^{\alpha\gamma n}2^n 2^{-n(1+\beta)} = C\sum_{n=0}^{\infty} 2^{(\alpha\gamma-\beta)n} < \infty.
\end{aligned}$$

Theorem by Kolmogorov-Chentsov

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$$\mathsf{E}[r(X_s, X_t)^\alpha] \leq C|t-s|^{1+\beta}, \qquad 0 \leq s, t \leq \tau.$$
Then there is a modification $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ with continuous paths.
- Proof: So, there C' with $\xi_n \leq C' 2^{-\gamma n}$ for all $n = 0, 1, \ldots$ Now let $m \in \{0, 1, \ldots\}$ and $r \in [2^{-m-1}, 2^{-m}] \cap D$. So, for some C'',
$$\begin{aligned} \sup\{r(X_s, X_t) : s, t \in D, |s-t| \leq r\} \\ &= \sup_{n \geq m} \{r(X_s, X_t) : s, t \in D_n, |s-t| \leq r\} \\ &\leq 2 \sum_{n \geq m} \xi_n \leq 2C' \sum_{n \geq m} 2^{-\gamma n} \leq C'' 2^{-\gamma(m-1)} \leq C'' r^\gamma. \end{aligned}$$

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$$\mathrm{E}[r(X_s, X_t)^\alpha] \leq C|t - s|^{1+\beta}, \qquad 0 \leq s, t \leq \tau.$$
Then there is a modification $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ with continuous paths.
- Proof: So, every path on D is Hölder-continuous to the parameter γ. This means that $\mathcal{X}$ can be extended Hölder-continuously to I. We call his continuous extension $\mathcal{Y} = (Y_t)_{t \in I}$. Now, let $t \in I$ $t_n \to t$ in D. Since $\mathrm{P}(r(X_{t_n}, X_t) > \varepsilon) \leq \mathrm{E}[r(X_{t_n}, X_t)^\alpha]/\varepsilon^\alpha \xrightarrow{n\to\infty} 0$, $X_{t_n} \xrightarrow{n\to\infty}_p X_t$. By continuity of $\mathcal{Y}$, we find $Y_{t_n} \xrightarrow{n\to\infty}_{fs} Y_t$. In particular, $\mathrm{P}(X_t = Y_t) = 1$.