universität freiburg

Stochastic processes — Winter semester 2024

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 5 - Martingales I

Exercise 1 (2+2=4 Points).

(a) Let $X \sim N(\mu,\sigma^2)$ with $\mu \neq 0$ and $\sigma^2 > 0$. Prove that there is a unique $\theta \neq 0$ such that $\mathbf{E}[e^{\theta X}] = 1$.

(b) Let $(X_i)_{i=1,2,\dots}$ be iid with $X_0 \sim N(\mu,\sigma^2)$ with $\mu \neq 0$ and $\sigma^2 > 0$. Show that $\mathcal{Z} = (Z_n)_{n=0,1,2,\dots}$ with $Z_n = e^{\theta \sum_{j=1}^n X_j}$ is a martingale with θ defined in (a).

Exercise 2 (4 points).
Let $\mathcal{X} = (X_n)_{n\geq 0}$ be a supermartingale with respect to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$. Show the following: $\mathcal{X}$ is a martingale iff there exists a sequence $(n_m)_{m\geq 1}$ with $n_m \xrightarrow{m\to\infty} \infty$ and $\mathbf{E}[X_{n_m}] \geq \mathbf{E}[X_0]$.

Exercise 3 (2+2=4 Points).
Let $\mathcal{X} = (X_t)_{t\geq 0}$ and $\mathcal{Y} = (Y_t)_{t\geq 0}$ be square integrable stochastic processes, adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$. They are said to be conditionally uncorrelated, if

$$\mathbf{E}[(X_t - X_s)(Y_t - Y_s)|\mathcal{F}_s] = 0, \qquad 0 \leq s \leq t < \infty.$$

(a) Give an example of non-conditionally uncorrelated $\mathcal{X}$ and $\mathcal{Y}$.

(b) Show that $(X_t Y_t)_{t\geq 0}$ is a martingale iff $\mathcal{X}$ and $\mathcal{Y}$ are conditionally uncorrelated.

Exercise 4 (2+2=4 points).
Let $\mathcal{B} = (B_t)_{t\geq 0}$ be a Brownian Motion, started in $B_0 = 0$. For a constant $a > 0$ define $T := \inf\{t \geq 0 : B_t \notin (-a,a)\}$.

(a) Why is T a stopping time with respect to $\mathcal{F} = \{\mathcal{F}_t^{\mathcal{B}}\}_{t\geq 0}$.

(b) Show that, for all c,

$$X_t := \exp\left(-\frac{c^2}{2}t\right)\cosh(cB_t)$$

defines a martingale $(X_t)_{t\geq 0}$ with respect to $\{\mathcal{F}_t^{\mathcal{B}}\}_{t\geq 0}$.