universität freiburg

Stochastic processes

Winter semester 2024

Lecture: Prof. Dr. Peter Pfaffelhuber
Assistance: Samuel Adeosun
https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 4 - Filtrations and stopping times

In all exercises, we have a probability space $(\Omega,\mathcal{F},\mathbf{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$.

Exercise 1 (4 points).
Verify that

$$\mathcal{F}_T := \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t, t \in I\}$$

is a σ-algebra.

Exercise 2 (4 Points).
Let $\Omega = [0,1]$ and $\mathcal{F} = \mathcal{B}([0,1])$. Define a stochastic process $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$ by

$$X_n(\omega) := 2\omega 1\!\!1_{[0,1-\frac{1}{n}]}(\omega).$$

Show that the generated filtration $(\mathcal{F}_n^{\mathcal{X}})_{n\in\mathbb{N}}$ is given by

$$\mathcal{F}_n^{\mathcal{X}} = \left\{ A \cup B : A \in \mathcal{B}((0,1-\frac{1}{n}]), \, B \in \{\emptyset,\{0\} \cup (1-\frac{1}{n},1]\} \right\}$$

Exercise 3 (4 points).
Given an optional time T of the filtration $(\mathcal{F}_t)_{t\in I}$, consider the sequence $(T_n)_{n\geq 1}$ of random times given by

$$T_n(\omega) = \begin{cases} T(\omega); & \text{on } \{\omega;\ T(\omega) = +\infty\} \\ \\ \frac{k}{2^n}; & \text{on } \left\{\omega;\ \frac{k-1}{2^n} \leq T(\omega) < \frac{k}{2^n}\right\} \end{cases}$$

for $n \geq 1, k \geq 1$. Clearly, $T_n \geq T_{n+1} \geq T$, for $n \geq 1$. Show that each T_n is a stopping time, and that $\lim_{n\to\infty} T_n = T$.

Exercise 4 (2+2 Points).

1. Let $(X_t)_{t\in[0,\infty)}$ be a stochastic process, such that, for all $t \geq 0$,

   $$Y_t : \begin{cases} I \cap [0,t] \times \Omega & \to E \\ (s,\omega) & \mapsto X_s(\omega) \end{cases}$$

   is measurable with respect to $I \cap \mathcal{B}([0,t]) \otimes \mathcal{F}_t/\mathcal{B}(E)$. Show that $(X_t)_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$.

We call $(X_t)_{t\geq 0}$ measurable if

$$\begin{cases} I \times \Omega & \to E \\ (s,\omega) & \mapsto X_s(\omega) \end{cases}$$

is measurable with respect to $\mathcal{B}([0,\infty)) \otimes \mathcal{F}/\mathcal{B}(E)$. Show that every progressively measurable process is measurable, but need not even be adapted if it is only measurable.

2. Let $(\mathcal{X}_t)_{t\geq 0}$ be a stochastic process, which is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$ and T be an $(\mathcal{F}_t)_{t\geq 0}$ stopping time. Show that the stopped process $(X_{T\wedge t})_{t\geq 0}$ is also progressively measurable.