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## Stochastic processes **Winter semester 2024**

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
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# Tutorial 3 - Poisson process and Brownian motion

**Exercise 1** (2+2 points).
Assume that the service of buses in Freiburg starts at 8 pm and then they arrive according to a Poisson process of intensity $\lambda = 4$ per hour. Franz Kafka starts to wait for a bus at 8pm.

(a) What is the expected waiting time for the next bus?

(b) At 8:30pm Kafka is still waiting. What is now the expected waiting time?

*Solution.*
Suppose $t = 0$ corresponds to the time 8pm and let $\mathcal{X} = (X_t)_{t\geq 0}$ denote the Poisson process with intensity $\lambda = 4$. According to Proposition 13.13, we can assume that there exists independent, exponentially distributed random variables $S_1, S_2, \dots$ with parameter $\lambda$ such that

$$X_t := \begin{cases} 0 & \text{if } t = 0, \\ \max\{k \in \{0,1,2,3,\dots\} : S_1 + \dots + S_k \leq t\}, & \text{if } t > 0. \end{cases}$$

(a) We can then model the expected waiting time by $S_1$ and $\mathbf{E}[S_1] = \frac{1}{\lambda} = 15\,\text{min}$.

(b) Let $Y$ model the waiting time for the next bus given that Kafka has already waited for 30min. By the definition of conditional probability, we have that for each $t \geq 0$,

$$\begin{aligned}\mathbf{P}(Y > t) &= \mathbf{P}(S_1 > 0.5 + t | S_1 > 0.5) \\ &= \mathbf{P}(S_1 > 0.5 + t)\mathbf{P}(S > 0.5) = e^{-4(0.5+t)}e^{-4(0.5)} = e^{-4t}.\end{aligned}$$

Hence, $Y$ is also exponentially distributed with the same parameter. This implies that Franz Kafka is in fact not in a better situation than at 8pm, as the expected waiting time is again $\mathbf{E}[Y] = \frac{1}{\lambda} = 15\,\text{min}$.

**Exercise 2** (1+3 Points).

(a) Let $Z$ be a standard normal random variable. For all $t \geq 0$, let $X_t = \sqrt{t}Z$. The stochastic process $\mathcal{X} = \{X_t : t \geq 0\}$ has continuous paths and $\forall t \geq 0$, $X_t \sim N(0,t)$. Is $\mathcal{X}$ a Brownian motion? Justify!

(b) Let $W_t$ and $\tilde{W}_t$ be two independent Brownian motion and $\rho$ is a constant contained in the unit interval. For all $t \geq 0$, let $X_t = \rho W_t + \sqrt{1-\rho^2}\tilde{W}_t$. The stochastic process $\mathcal{X} = \{X_t : t \geq 0\}$ has continuous paths and $\forall t \geq 0$, $X_t \sim N(0,t)$. Is $\mathcal{X}$ a Brownian motion? Justify!

*Solution.*

(a) We check the stationary increments property for the process $\mathcal{X}$. That is; consider the increment for $0 \leq s < t$, we ascertain whether or not the distribution of the increments depends only on the time difference $t - s$.

$$X_t - X_s = \sqrt{t}Z - \sqrt{s}Z = (\sqrt{t} - \sqrt{s})Z.$$

Clearly,

$$\mathbf{E}[X_t - X_s] = (\sqrt{t} - \sqrt{s})\mathbf{E}[Z] = 0.$$

and,

$$\mathbf{Var}(X_t - X_s) = \mathbf{Var}((\sqrt{t} - \sqrt{s})Z) = (\sqrt{t} - \sqrt{s})^2\mathbf{Var}(Z) = (\sqrt{t} - \sqrt{s})^2.$$

The variance $(\sqrt{t} - \sqrt{s})^2$ does not equal $t - s$, indicating that the increments are not stationary. Thus, the process $\mathcal{X}$ is not a Brownian motion.

(b) We will first show that the increments $X_{t_i} - X_{t_{i-1}}$ are independent and that $X_{t_i} - X_{t_{i-1}} \sim N(0,t_i - t_{i-1})$ for $i = 1, \ldots ,n$. We proceed as follows:

(i) **Independence of Increments.** Consider the increment: For $0 \leq t_1 < t_2 < \ldots < t_n$,

$$X_{t_i} - X_{t_{i-1}} = \rho(W_{t_i} - W_{t_{i-1}}) + \sqrt{1-\rho^2}(\tilde{W}_{t_i} - \tilde{W}_{t_{i-1}}).$$

The increments $W_{t_i} - W_{t_{i-1}}$ and $\tilde{W}_{t_i} - \tilde{W}_{t_{i-1}}$ are independent because they are increments of independent Brownian motions. Furthermore, for $i \neq j$, the increments $X_{t_i} - X_{t_{i-1}}$ and $X_{t_j} - X_{t_{j-1}}$ involve disjoint intervals of the Brownian motions (i.e., the increments do not overlap). Thus, since increments of independent processes are independent, it follows that:

$$X_{t_i} - X_{t_{i-1}} \text{ is independent of } X_{t_j} - X_{t_{j-1}} \text{ for } i \neq j.$$

(ii) **Distribution of Increments**. We show that $X_{t_i} - X_{t_{i-1}} \sim N(0,t_i - t_{i-1})$. The mean is:

$$\mathbf{E}[X_{t_i} - X_{t_{i-1}}] = \mathbf{E}[\rho(W_{t_i} - W_{t_{i-1}})] + \mathbf{E}[\sqrt{1-\rho^2}(\tilde{W}_{t_i} - \tilde{W}_{t_{i-1}})] = 0.$$

The variance is computed as follows:

$$\mathbf{Var}(X_{t_i} - X_{t_{i-1}}) = \mathbf{Var}\left(\rho(W_{t_i} - W_{t_{i-1}}) + \sqrt{1-\rho^2}(\tilde{W}_{t_i} - \tilde{W}_{t_{i-1}})\right).$$

Since $W_{t_i} - W_{t_{i-1}}$ and $\tilde{W}_{t_i} - \tilde{W}_{t_{i-1}}$ are independent, we have:

$$\mathbf{Var}(X_{t_i} - X_{t_{i-1}}) = \rho^2\mathbf{Var}(W_{t_i} - W_{t_{i-1}}) + (1-\rho^2)\mathbf{Var}(\tilde{W}_{t_i} - \tilde{W}_{t_{i-1}}).$$

Since $\mathbf{Var}(W_{t_i} - W_{t_{i-1}}) = t_i - t_{i-1}$ and $\mathbf{Var}(\tilde{W}_{t_i} - \tilde{W}_{t_{i-1}}) = t_i - t_{i-1}$, we have:

$$\mathbf{Var}(X_{t_i} - X_{t_{i-1}}) = \rho^2(t_i - t_{i-1}) + (1-\rho^2)(t_i - t_{i-1}) = (t_i - t_{i-1})(\rho^2 + (1-\rho^2)) = t_i - t_{i-1}.$$

Thus,

$$X_{t_i} - X_{t_{i-1}} \sim N(0,t_i - t_{i-1}).$$

**Exercise 3** (2+2=4 Points)**.**
Let $d,k \in \mathbb{N}$, $C \in R^{d\times d}$ and $X \sim N(0,C)$. (Recall from definition 10.14 from the manuscript of probability theory!)

(a) If there is $S \in \mathbb{R}^{k\times d}$ with $C = S^\top S$, and $Z \sim N(0,I_k)$ (where $I_k$ is the $k \times k$ identity matrix), show that $S^\top Z \overset{d}{=} X$.

(b) Let $t_1 \le ... \le t_n$ and $Z \sim N(0,I_d)$. Find $S \in \mathbb{R}^{d\times d}$ such that $S^\top Z \sim N(0,C)$ with $C_{ij} := t_i \wedge t_j$ (as in the covariance matrix of Brownian Motion).

*Solution.*

(a) We have

$$\mathbf{E}[S^\top Z] = S^\top \mathbf{E}[Z] = S^\top \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0.$$

Also,

$$\begin{aligned}
\mathbf{Cov}(S^\top Z,S^\top Z) &= \mathbf{E}[(S^\top Z)(S^\top Z)^\top] - \underbrace{\mathbf{E}[S^\top Z]\mathbf{E}[S^\top Z]^\top}_{=0} \\
&= \mathbf{E}[S^\top Z Z^\top S] \quad \text{(expanding the first term)} \\
&= S^\top \mathbf{E}[Z Z^\top] S \quad \text{(using linearity of expectation)} \\
&= S^\top I_d S \quad (\text{since } \mathbf{E}[Z Z^\top] = I_d) \\
&= S^\top S \quad \text{(since multiplying by the identity does not change the matrix)} \\
&= C.
\end{aligned}$$

Since both $S^\top Z$ and $X$ are distributed as $N(0,C)$, where the distribution of $S^\top Z$ is uniquely determined by the expected value and the covariance matrix, we have $S^\top Z \overset{d}{=} X$.

(b) The covariance matrix $C$ is defined as $C_{ij} = t_i \wedge t_j$, which is symmetric and positive semi-definite, representing the covariance structure of Brownian motion. To show that $C$ is symmetric, we note:$C_{ij} = t_i \wedge t_j = t_j \wedge t_i = C_{ji}$. To show that $C$ is positive semi-definite, we need to prove: $x^\top C x \ge 0 \quad \text{for all } x \in \mathbb{R}^d$. Expanding this expression gives:

$$x^\top C x = \sum_{i=1}^{d} \sum_{j=1}^{d} x_i x_j (t_i \wedge t_j).$$

Since $t_i \wedge t_j$ represents the minimum of two non-negative terms, we have:

$$x^\top C x = \sum_{i=1}^{d} \sum_{j=1}^{d} x_i x_j (t_i \wedge t_j) \ge 0.$$

Now because $C$ is symmetric and positive semi-definite, we can perform a Cholesky decomposition: $C = LL^\top$, where $L$ is a lower triangular matrix. To satisfy the condition $\mathbf{Cov}(S^\top Z,S^\top Z) = C$, we set: $S = L^\top$, making $S$ an upper triangular matrix. Since $Z \sim N(0,I_d)$

$$S^\top Z = (L^\top)^\top Z = LZ \sim N(0,LZZ^\top L^\top) = N(0,LI_dL^\top) = N(0,C).$$

Thus, the matrix $S$ that satisfies $S^\top Z \sim N(0,C)$ is given by:

$$S = L^\top \quad \text{where } C = LL^\top \text{ and } C_{ij} = t_i \wedge t_j.$$

**Exercise 4** (4 Points)**.**
Let $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$ be a Gaussian process. Show that if $\mathcal{X}$ converges to a random variable $X$ in probability, it also converges in $\mathcal{L}^2$ to $X$.

*Solution.*
We can use one of the results from probability theory here. (See Theorem 7.11!) First, Chebyshev's inequality allows us for $\varepsilon > 0$,

$$\mathbf{P}(|X_n - X| \geq \varepsilon) \leq \frac{\mathbf{E}[|X_n - X|^2]}{\varepsilon^2}.$$

Recall that $\mathcal{X}$ is called *Gaussian* if $c_1X_{t_1} + \cdots + c_nX_{t_n}$ for each choice of $c_1,...,c_n \in \mathbb{R}$ and $t_1,...,t_n \in I$ is normally distributed. Since $X_n \xrightarrow{n\to\infty}_p X$, according to proposition 7.6 there is a subsequence $n_1,n_2,\dots$ with $X_{n_k} \xrightarrow{k\to\infty} X$ almost surely. With Fatou's Lemma,

$$\mathbf{E}[|X|^2] = \mathbf{E}[\liminf_{k\to\infty} |X_{n_k}|^2] \leq \sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^2] < \infty$$

because of Lemma 7.9. In particular, $X \in \mathcal{L}^2$. For every $\delta > 0$, due to convergence in probability,

$$\mathbf{P}(|X_n - X| > \delta) \xrightarrow{n\to\infty} 0.$$

From lemma 7.9 and dominated convergence,

$$\lim_{n\to\infty} \mathbf{E}[|X_n - X|^2] = \lim_{n\to\infty} \mathbf{E}[|X_n - X|^2; |X_n - X| > \delta] + \mathbf{E}[|X_n - X|^2; |X_n - X| \leq \delta] \leq \delta^2.$$

Since $\delta > 0$ was arbitrary, $X_n \xrightarrow{n\to\infty}_{\mathcal{L}^2} X$ follows.