universität freiburg

Stochastic processes **Winter semester 2024**

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 3 - Poisson process and Brownian motion

Exercise 1 (2+2 points)**.**
Assume that the service of buses in Freiburg starts at 8 pm and then they arrive according to a Poisson process of intensity $\lambda = 4$ per hour. Franz Kafka starts to wait for a bus at 8pm.

(a) What is the expected waiting time for the next bus?

(b) At 8:30pm Kafka is still waiting. What is now the expected waiting time?

Exercise 2 (1+3 Points)**.**

(a) Let Z be a standard normal random variable. For all $t \geq 0$, let $X_t = \sqrt{t}Z$. The stochastic process $\mathcal{X} = \{X_t : t \geq 0\}$ has continuous paths and $\forall t \geq 0$, $X_t \sim N(0,t)$. Is $\mathcal{X}$ a Brownian motion? Justify!

(b) Let W_t and $\tilde{W}_t$ be two independent Brownian motion and ρ is a constant contained in the unit interval. For all $t \geq 0$, let $X_t = \rho W_t + \sqrt{1-\rho^2}\tilde{W}_t$. The stochastic process $\mathcal{X} = \{X_t : t \geq 0\}$ has continuous paths and $\forall t \geq 0$, $X_t \sim N(0,t)$. Is $\mathcal{X}$ a Brownian motion? Justify!

Exercise 3 (2+2=4 Points)**.**
Let $d,k \in \mathbb{N}$, $C \in R^{d\times d}$ and $X \sim N(0,C)$. (Recall from definition 10.14 from the manuscript of probability theory!)

(a) If there is $S \in \mathbb{R}^{k\times d}$ with $C = S^\top S$, and $Z \sim N(0,I_k)$ (where I_k is the $k \times k$ identity matrix), show that $S^\top Z \overset{d}{=} X$.

(b) Let $t_1 \leq ... \leq t_n$ and $Z \sim N(0,I_d)$. Find $S \in \mathbb{R}^{d\times d}$ such that $S^\top Z \sim N(0,C)$ with $C_{ij} := t_i \wedge t_j$ (as in the covariance matrix of Brownian Motion).

Exercise 4 (4 Points)**.**
Let $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$ be a Gaussian process. Show that if $\mathcal{X}$ converges to a random variable X in probability, it also converges in $\mathcal{L}^2$ to X.