universität freiburg

Stochastic processes

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 2 - Definition and existence of stochastic processes

Exercise 1 (1+2+1=4 Points)**.**
Let $\Omega = \{1,2,3,4,5\}$.

(a) Find the smallest σ- algebra $\mathcal{F}_1$ containing

$$\mathcal{F}_2 := \{\{1,2,3\},\{3,4,5\}\}.$$

(b) Is the random variable $X : \Omega \to \mathbb{R}$ defined by

$$X(1) = X(2) = 0, \quad X(3) = 10, \quad X(4) = X(5) = 1$$

measurable with respect to $\mathcal{F}_1$?

(c) Find the σ-algebra $\mathcal{F}_3$ generated by $Y : \Omega \to \mathbb{R}$ and defined by

$$Y(1) = 0, \quad Y(2) = Y(3) = Y(4) = Y(5) = 1.$$

Exercise 2 (2+2 points)**.**

1. Given an example of two stochastic processes $\mathcal{X}$ and $\mathcal{Y}$ which are versions of each other, but no modifications of each other.
2. Give an example of a real-valued stochastic process $\mathcal{X}$, such that $\mathbb{V}[X_t] > 0$ for all t and $\mathcal{X} = (X_t)_{t\in I}$ and $\mathcal{X}^2 := (X_t^2)_{t\in I}$ are indistinguishable.

Exercise 3 (4 Points)**.**
Let I be some index set, (E,r) be Polish and $(\mathbf{P}_i)_{i\in I}$ a family of probability measures on $\mathcal{B}(E)$. Show that there exists an E-valued stochastic process $(X_t)_{t\in I}$ such that $(X_{t_1},\dots,X_{t_n}) \sim \otimes_{i=1}^n \mathbf{P}_{t_i}$ for any $t_1,\dots,t_n \in I$. In other words, $(X_t)_{t\in I}$ is an independent family with $X_t \sim \mathbf{P}_t$.

Exercise 4 (1+1+2= 4 Points)**.**
Let $(X_k)_{k\in\mathbb{N}_0}$ be a **symmetric** simple random walk, i.e. $X_k = \sum_{i=0}^{k-1} Z_i$ where $Z_1,Z_2,\dots$ are iid with $\mathbf{P}(Z_1 = \pm 1) = \frac{1}{2}$[1]. For $n \in \mathbb{N}$ define

$$S_n = \sum_{k=1}^{n} X_k.$$

[1] Recall the convention that $\sum_{i=0}^{-1} a_i = 0$

(a) Show whether or not $(S_n)_{n\in\mathbb{N}_0}$ is a simple random walk (not necessarily symmetric).

(b) Compute the covariance $\mathbf{COV}[X_k, X_l]$ for $k \le l \in \mathbb{N}$.

(c) Compute the variance of S_n for $n \in \mathbb{N}$.

Note: You may need to recall that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \text{for } n \in \mathbb{N}.$$