universität freiburg

## Stochastic processes **Winter semester 2024**

Lecture: Prof. Dr. Peter Pfaffelhuber
Assistance: Samuel Adeosun
https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

# Tutorial 1 - Repetition of probability theory

**Exercise 1** (2+2 =4 Points)**.**
Let $X \geq 0$ be a nonnegative real-valued random variable.

(a) Assume that $\mathbf{E}[X] < \infty$. Show that $n\mathbf{E}\left[\ln\left(1 + \frac{X}{n}\right)\right] \to \mathbf{E}[X]$ as $n \to \infty$.

(b) Assume that $\mathbf{E}[X] = \infty$. Show that $n\mathbf{E}\left[\ln\left(1 + \frac{X}{n}\right)\right] \to \infty$ as $n \to \infty$.

*Hint:* It might be helpful to show that $n \mapsto (1 + \frac{X}{n})^n$ is increasing.

*Solution.*

(a) From

$$(1 + \tfrac{X}{n})^n = \sum_{k=0}^n \binom{n}{k} (\tfrac{X}{n})^k = \sum_{k=0}^n \frac{X^k}{k!} \frac{n \cdots (n-k+1)}{n^k}$$

we see the monotonicity. Set $X_n = n \ln\left(1 + \frac{X}{n}\right)$. We also note that for $x \geq 0$, the inequality $\ln(1+x) \leq x$ holds. Thus, $\ln\left(1 + \frac{X}{n}\right) \leq \frac{X}{n}$. That is, $n \ln\left(1 + \frac{X}{n}\right) \leq X$ (an integrable random variable which does not depend on $n$. Taking expectations, we have:

$$n\mathbf{E}\left[\ln\left(1 + \frac{X}{n}\right)\right] \leq \mathbf{E}[X].$$

Next, we analyze the limit of $n \ln\left(1 + \frac{X}{n}\right)$ as $n \to \infty$. We can use the fact that:

$$\lim_{n\to\infty} n \ln\left(1 + \frac{X}{n}\right) = X.$$

By the Dominated Convergence Theorem, since $X$ is integrable and dominates $n \ln\left(1 + \frac{X}{n}\right)$, we conclude:

$$\lim_{n\to\infty} \mathbf{E}\left[n \ln\left(1 + \frac{X}{n}\right)\right] = \mathbf{E}[X].$$

Thus, we have:

$$n\mathbf{E}\left[\ln\left(1 + \frac{X}{n}\right)\right] \to \mathbf{E}[X] \quad \text{as } n \to \infty.$$

(b) By applying Fatou's lemma, we have:

$$\liminf_{n\to\infty} \mathbf{E}\left[n\ln\left(1+\frac{X}{n}\right)\right] \geq \mathbf{E}\left[\liminf_{n\to\infty} n\ln\left(1+\frac{X}{n}\right)\right].$$

From part (a), we know that:

$$\lim_{n\to\infty} n\ln\left(1+\frac{X}{n}\right) = X \implies \liminf_{n\to\infty} n\ln\left(1+\frac{X}{n}\right) = X.$$

Since $\mathbf{E}[X] = \infty$, it follows that:

$$\liminf_{n\to\infty} \mathbf{E}\left[n\ln\left(1+\frac{X}{n}\right)\right] \geq \mathbf{E}[X] = \infty.$$

Hence, we conclude that:

$$n\mathbf{E}\left[\ln\left(1+\frac{X}{n}\right)\right] \to \infty \quad \text{as } n\to\infty.$$

**Exercise 2** (3+1= 4 Points)**.**

(a) Let $(X_n)_{n\in\mathbb{N}}$ be an independent family of random variables such that $\mathbf{P}(X_n = -1) = \mathbf{P}(X_n = +1) = \frac{1}{2}$ and let $S_n = X_1 + X_2 + \ldots + X_n$ for any $n \in \mathbb{N}$. Show that $\limsup_{n\to\infty} S_n = \infty$ almost surely.

(b) Suppose that $X$ and $Y$ are random variables in $\mathcal{L}^2$ such that

$$\mathbf{E}[X|Y] = Y \quad \text{and} \quad \mathbf{E}[Y|X] = X \quad \text{almost surely.}$$

Show that $X = Y$ almost surely.

*Hint:* Theorem 11.2

*Solution.*

(a) Recall that the Kolmogorov's 0-1 law helps to clarify when sums of independent random variables are almost sure to converge. We can set $\mathcal{F}_i := \sigma(X_i)$, $i = 1,2,\ldots$. Since the $(X_n)_{n\in\mathbb{N}}$ are independent, the above means that we have an independent family of $\sigma$-algebras $(\mathcal{F}_i)_{i=1,2,\ldots}$. Using the Kolmogorov's 0-1 law, we know that $\mathcal{F}_1,\mathcal{F}_2,\dots \subseteq \mathcal{F}$ independent implies $\mathcal{T} := \mathcal{T}(\mathcal{F}_1,\mathcal{F}_2,\ldots) = \mathcal{T}((\sigma(X_1),\sigma(X_2),\ldots)$ $\mathbf{P}$-trivial. By definition,

$$\mathcal{T}(\mathcal{F}_1,\mathcal{F}_2,\ldots) = \bigcap_{n\geq 1} \sigma\Big(\bigcup_{m>n} \mathcal{F}_m\Big) \quad \text{and} \quad \limsup_{n\to\infty} S_n := \bigcap_{n\geq 1}\bigcup_{m\geq n} S_m,$$

with $S_m = X_m + X_{m+1} + \ldots + X_k$ measurable with respect to $\sigma\Big(\bigcup_{m\geq n}\mathcal{F}_m\Big) = \sigma\Big(\bigcup_{m\geq n}\sigma(X_m)\Big)$. Using the symmetric property of the random variables, $\limsup_{n\to\infty} S_n = \infty$ and $\liminf_{n\to\infty} S_n = -\infty$ have the same probability which is either 0 or 1. From the above, we could not have $\mathbf{P}(\limsup_{n\to\infty} S_n) = 0$, hence, $\mathbf{P}(\limsup_{n\to\infty} S_n) = 1$ and so $\limsup_{n\to\infty} S_n = \infty$ almost surely.

(b) Define the difference $Z = X - Y$. We need to show that $Z = 0$ almost surely. From the first condition, we have:

$$\mathbf{E}[Z|Y] = \mathbf{E}[X - Y|Y] = \mathbf{E}[X|Y] - Y = Y - Y = 0.$$

Using the second given condition, we have

$$\mathbf{E}[Z|X] = \mathbf{E}[X - Y|X] = \mathbf{E}[X|X] - \mathbf{E}[Y|X] = X - X = 0.$$

It remains for us to ascertain that $\mathbf{E}[Z^2] = 0$. Consider the following:

$$\begin{aligned}\mathbf{E}[(X-Y)^2] &= \mathbf{E}[\mathbf{E}[(X-Y)^2|X]] = \mathbf{E}[\mathbf{E}[X^2 - 2XY + Y^2|X]] \\ &= \mathbf{E}[\mathbf{E}[X^2] - 2\mathbf{E}[XY] + \mathbf{E}[Y^2]|X] \\ &= \mathbf{E}[X^2 - 2X\underbrace{\mathbf{E}[Y|X]}_{X} + \mathbf{E}[Y^2|X]] \\ &= \mathbf{E}[X^2] - \mathbf{E}[Y^2] = \mathbf{E}[Y^2 - X^2]\end{aligned}$$

If we condition on $Y$ instead,

$$\begin{aligned}\mathbf{E}[(X-Y)^2] &= \mathbf{E}[\mathbf{E}[(X-Y)^2|Y]] = \mathbf{E}[\mathbf{E}[X^2 - 2XY + Y^2|Y]] \\ &= \mathbf{E}[\mathbf{E}[X^2] - 2\mathbf{E}[XY] + \mathbf{E}[Y^2]|Y] \\ &= \mathbf{E}[X^2 - 2Y\underbrace{\mathbf{E}[X|Y]}_{Y} + \mathbf{E}[Y^2]] \\ &= \mathbf{E}[X^2] - \mathbf{E}[Y^2] = \mathbf{E}[X^2 - Y^2]\end{aligned}$$

That is, we get

$$\mathbf{E}[Z^2] = \mathbf{E}[(X-Y)^2] = \mathbf{E}[Y^2 - X^2] = \mathbf{E}[X^2 - Y^2] = 0$$

Therefore, $X = Y$ almost surely.

**Exercise 3** (2+2=4 Points)**.**
For every $n \in \mathbb{N}$, let $X_n$ be a random variable with probability density

$$f_X(x) = nx^{-n-1}\mathbb{1}_{[1,\infty)}(x).$$

(a) Determine the distribution function of $X_n$ and show that $X_n \xrightarrow{n\to\infty}_p 1$.

(b) Investigate for which $n$ the expected value of $X_n$ exists and, if necessary, specify it. Does the convergence from (a) also apply in $\mathcal{L}^1$?

*Solution.*

(a) For $x > 1$, $F_{X_\alpha}(x) = \int_1^x \alpha t^{-\alpha-1}dt = -t^{-\alpha}\Big|_{t=1}^{x} = 1 - x^{-\alpha}$ and $F_{X_\alpha}(x) = 0$ otherwise. Thus, it follows that for $\varepsilon > 0$,

$$\mathbf{P}(|X_\alpha - 1| > \varepsilon) = \mathbf{P}(X_\alpha > 1 + \varepsilon) = (1+\varepsilon)^{-\alpha} \to 0.$$

(b) We have

$$\mathbf{E}[X_\alpha] = \int_1^\infty \alpha x^{-\alpha} dx = \begin{cases} \log x \Big|_1^\infty & = \infty \text{ for } \alpha = 1, \\ \left(\frac{\alpha}{1-\alpha} x^{1-\alpha}\right)\Big|_1^\infty & = \begin{cases} \infty & \text{for } \alpha < 1 \\ \frac{\alpha}{\alpha-1} & \text{for } \alpha > 1. \end{cases} \end{cases}$$

It also applies for $\alpha > 1$ that

$$\|X_\alpha - 1\|_{\mathcal{L}^1} = \mathbf{E}[X_\alpha] - 1 = \frac{\alpha}{\alpha - 1} - 1 = \frac{1}{\alpha - 1} \to 0.$$

**Exercise 4** (1+1+1+1= 4 Points)**.**

You roll two six-sided fair dice, where one die has the digits $\{1,2,3,4,5,6\}$ and the second die has the digits $\{3,3,6,6,6,6\}$. Let $X_i, i = 1,2$ be the result of the $i$th roll and $Y = X_1 + X_2$.

(a) Enter the probabilities $\mathbf{P}(Y = k)$ for $k = 2,3,...,12$ in the table.

| $k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{P}(Y = k)$ | | | | | | | | | | | |

(b) Determine the expected value of $Y$.

(c) Enter the conditional probabilities $\mathbf{P}(X_2 = x|Y = k)$ for $x = 3,6$ and $k = 2,3,...,12$ in the table.

| $k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{P}(X_2 = 3\|Y = k)$ | | | | | | | | | | | |
| $\mathbf{P}(X_2 = 6\|Y = k)$ | | | | | | | | | | | |

(d) Determine the conditional expectations $\mathbf{E}[X_1|Y]$ and $\mathbf{E}[X_2|Y]$.

*Solution.*

(a) We have $\{Y = k\} = (\{X_2 = 6\} \cap \{X_1 = k - 6\}) \cup (\{X_2 = 3\} \cap \{X_1 = k - 3\})$.

For $k = 2,3$, $\{X_1 = k - 3\} = \{X_1 = k - 6\} = \emptyset$ and thus $\mathbf{P}(Y = k) = 0$.

For $k = 4,5,6$, $\{X_1 = k - 6\} = \emptyset$ and therefore

$$\mathbf{P}(Y = k) = \mathbf{P}(X_1 = k - 3)\mathbf{P}(X_2 = 3) = \tfrac{1}{6} \cdot \tfrac{1}{3} = \tfrac{1}{18}.$$

For $k = 10,11,12$, $\{X_1 = k-3\} = \emptyset$ and therefore

$$\mathbf{P}(Y=k) = \mathbf{P}(X_1 = k-6)\mathbf{P}(X_2 = 6) = \tfrac{1}{6} \cdot \tfrac{2}{3} = \tfrac{1}{9}.$$

For $k = 7,8,9$ it is the case that

$$\mathbf{P}(Y=k) = \mathbf{P}(X_1 = k-6)\mathbf{P}(X_2 = 6) + \mathbf{P}(X_1 = k-3)\mathbf{P}(X_2 = 3) = \tfrac{1}{6} \cdot \tfrac{2}{3} + \tfrac{1}{6} \cdot \tfrac{1}{3} = \tfrac{1}{6}.$$

| $k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{P}(Y=k)$ | 0 | 0 | $\frac{1}{18}$ | $\frac{1}{18}$ | $\frac{1}{18}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{1}{9}$ |

(b) There are two possibilities for the expected value:

- Calculate with the probabilities $\mathbf{P}(Y=k)$ that

  $\mathbf{E}[Y] = \frac{1}{18}(4+5+6) + \frac{1}{6}(7+8+9) + \frac{1}{9}(10+11+12) = \frac{5}{6} + \frac{24}{6} + \frac{22}{6} = \frac{17}{2} = 8.5.$

- Alternatively, calculate $\mathbf{E}[X_1] = \frac{1}{6}(1 + ... + 6) = 3.5$, $\mathbf{E}[X_2] = 3 \cdot \frac{1}{3} + 6 \cdot \frac{2}{3} = 5$ and with linearity of the expected value

$$\mathbf{E}[Y] = 3.5 + 5 = 8.5.$$

(c) On the set $Y \in \{4,5,6\}$ is $X_2 = 3$, on the set $Y \in \{10,11,12\}$ is $X_2 = 6$. For $k \in \{7,8,9\}$ we have

$$\mathbf{P}(X_2 = 3|Y=k)) = \mathbf{P}(X_2 = 3, Y = k)/\mathbf{P}(Y=k)\mathbf{P}(X_2 = 3, X_1 = k-3)/\mathbf{P}(Y=k) = \tfrac{1}{3} \cdot \tfrac{1}{6}/\tfrac{1}{6} = \tfrac{1}{3}$$

and analogously $\mathbf{P}(X_2 = 6|Y=k)) = \frac{2}{3}$.

| $k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{P}(X_2 = 3\|Y=k)$ | 0 | 0 | 1 | 1 | 1 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 | 0 |
| $\mathbf{P}(X_2 = 6\|Y=k)$ | 0 | 0 | 0 | 0 | 0 | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 1 | 1 | 1 |

(d) We first determine $\mathbf{E}[X_2|Y]$: The sets $\{\{Y=4\},...,\{Y=12\}\}$ are a partition of $\Omega$, therefore

$$\mathbf{E}[X_2|Y] = \sum_{k=4}^{12} \mathbf{E}[X_2|Y=k] \cdot 1_{Y=k} = 3 \cdot 1_{Y \in \{4,5,6\}} + \underbrace{\left(3 \cdot \tfrac{1}{3} + 6 \cdot \tfrac{2}{3}\right)}_{=5} \cdot 1_{Y \in \{7,8,9\}} + 6 \cdot 1_{Y \in \{10,11,12\}}$$

For $\mathbf{E}[X_1|Y]$ we obtain with linearity of the conditional expectation that

$$Y = \mathbf{E}[Y|Y] = \mathbf{E}[X_1|Y] + \mathbf{E}[X_2|Y]$$

and thus,

$$\mathbf{E}[X_1|Y] = Y - \mathbf{E}[X_2|Y] = (Y-3) \cdot 1_{Y \in \{4,5,6\}} + (Y-5) \cdot 1_{Y \in \{7,8,9\}} + (Y-6) \cdot 1_{Y \in \{10,11,12\}}.$$

**Alternative way to calculate $\mathbf{E}[X_2|Y]$:** We define

$$f(Y) = 3 \cdot 1_{Y\in\{4,5,6\}} + \mathbf{E}[X_2] \cdot 1_{Y\in\{7,8,9\}} + 6 \cdot 1_{Y\in\{10,11,12\}}$$

and use the definition of the conditional expectation to show that $\mathbf{E}[X_2|Y] = f(Y)$. Integrability and $\sigma(Y)$ measurability are clear. It must therefore be shown that for any $A \in \sigma(Y)$ it is true that

$$\mathbf{E}[X_2; A] = \mathbf{E}[f(Y); A].$$

It is sufficient to check this property on the generator $\{Y = k\}$,$k = 4,...,12$ of $\sigma(Y)$. For $k \in \{4,5,6,10,11,12\}$ this follows from $X_2 \cdot 1_{Y=k} = 3$ for $k = 4,5,6$, or $X_2 \cdot 1_{Y=k} = 6$ for $k = 10,11,12$. For $k = 7,8,9$ is

$$\begin{aligned}
\mathbf{E}[X_2; Y = k] &= \mathbf{E}[X_2; Y = k, X_1 = k - 6] + \mathbf{E}[X_2; Y = k, X_1 = k - 3] \\
&= \mathbf{E}[X_2; X_2 = 6, X_1 = k - 6] + \mathbf{E}[X_2; X_2 = 3, X_1 = k - 3] \\
&= 6\mathbf{P}(X_2 = 6, X_1 = k - 6) + 3\mathbf{P}(X_2 = 3, X_1 = k - 3) \\
&= 6 \cdot \tfrac{2}{3} \cdot \tfrac{1}{6} + 3 \cdot \tfrac{1}{3} \cdot \tfrac{1}{6} = 5 \cdot \tfrac{1}{6} \\
&= 5 \cdot \mathbf{P}(Y = k) = \mathbf{E}[f(Y); Y = k].
\end{aligned}$$