Stochastic processes Winter semester 2024

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https://pfaffelh.github.io/hp/2024ws_stochproc.html
https://www.stochastik.uni-freiburg.de/

Tutorial 1 - Repetition of probability theory

Exercise 1 (2+2 =4 Points)**.**
Let $X \geq 0$ be a nonnegative real-valued random variable.

(a) Assume that $\mathbf{E}[X] < \infty$. Show that $n\mathbf{E}\left[\ln\left(1+\frac{X}{n}\right)\right] \to \mathbf{E}[X]$ as $n \to \infty$.

(b) Assume that $\mathbf{E}[X] = \infty$. Show that $n\mathbf{E}\left[\ln\left(1+\frac{X}{n}\right)\right] \to \infty$ as $n \to \infty$.

Hint: It might be helpful to show that $n \mapsto (1+\frac{X}{n})^n$ is increasing.

Exercise 2 (3+1= 4 Points)**.**

(a) Let $(X_n)_{n\in\mathbb{N}}$ be an independent family of random variables such that $\mathbf{P}(X_n = -1) = \mathbf{P}(X_n = +1) = \frac{1}{2}$ and let $S_n = X_1 + X_2 + \ldots + X_n$ for any $n \in \mathbb{N}$. Show that $\limsup_{n\to\infty} S_n = \infty$ almost surely.

(b) Suppose that X and Y are random variables in $\mathcal{L}^2$ such that

$$\mathbf{E}[X|Y] = Y \quad \text{and} \quad \mathbf{E}[Y|X] = X \quad \text{almost surely.}$$

Show that $X = Y$ almost surely.

Hint: Theorem 11.2

Exercise 3 (2+2=4 Points)**.**
For every $n \in \mathbb{N}$, let X_n be a random variable with probability density

$$f_X(x) = nx^{-n-1}\mathbb{1}_{[1,\infty)}(x).$$

(a) Determine the distribution function of X_n and show that $X_n \xrightarrow{n\to\infty}_p 1$.

(b) Investigate for which n the expected value of X_n exists and, if necessary, specify it. Does the convergence from (a) also apply in $\mathcal{L}^1$?

Exercise 4 (1+1+1+1= 4 Points)**.**
You roll two six-sided fair dice, where one die has the digits {1,2,3,4,5,6} and the second die has the digits {3,3,6,6,6,6}. Let X_i,$i = 1$,2 be the result of the ith roll and $Y = X_1 + X_2$.

(a) Enter the probabilities $\mathbf{P}(Y = k)$ for $k = 2,3,\dots,12$ in the table.

k	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{P}(Y = k)$											

(b) Determine the expected value of Y.

(c) Enter the conditional probabilities $\mathbf{P}(X_2 = x|Y = k)$ for $x = 3,6$ and $k = 2,3,\dots,12$ in the table.

k	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{P}(X_2 = 3\|Y = k)$											
$\mathbf{P}(X_2 = 6\|Y = k)$											

(d) Determine the conditional expectations $\mathbf{E}[X_1|Y]$ and $\mathbf{E}[X_2|Y]$.