Measure Theory for Probabilists
1. Introduction

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Introduction

- Course in spring 2024 at the University of Freiburg
- All course materials online at
- Prerequisites: a course in basic probability (coin tossing, throwing dice, binomial distribution, normal distribution)
- Goal: Solid introduction to all modern probability theory, including weak limits, stochastic processes, etc.
- Interference: courses in advanced calculus (Analysis III) might also cover measure theory
- Next course: Probability theory (summer 2024), covering all forms of convergence of random variables, conditional expectation, martingales

Measure theory

- Sample space Ω; $A \subseteq \Omega$
- Assign some value $\mu(A) \in \mathbb{R}_+$ to as many subsets of A as possible, with a number of computation rules
  $\Rightarrow$ measure μ defined on a σ-algebra $\mathcal{F} \subseteq 2^\Omega$
  $\to$ 1. Set systems; 2. Set functions
- Make a weighted average of some $f : \Omega \to \mathbb{R}$ with respect to the measure μ.
  $\Rightarrow$ integral $\int f d\mu$
  Study the structure of the space of functions with finite integral
  $\to$ 3. Measurable functions and the integral; 4. $\mathcal{L}^p$-spaces
- All the same on product spaces $\Omega = \bigtimes_{i \in I} \Omega_i$
  $\to$ 5. Product spaces

Measure Theory for Probabilists
2. Semi-rings, rings and σ-fields

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Definition of some set-systems

- $\mathcal{C} \subseteq 2^\Omega$

$$\mathcal{C}\ \sigma\text{-field} \quad \Longrightarrow \quad \mathcal{C}\ \text{ring} \quad \Longrightarrow \quad \mathcal{C}\ \text{semi-ring.}$$

- Definition 1.1: Ω set, $\emptyset \neq \mathcal{H}, \mathcal{R}, \mathcal{F} \subseteq 2^\Omega$.
  - $\mathcal{H}$ $\cap$-stable, if $(A, B \in \mathcal{H} \Rightarrow A \cap B \in \mathcal{H})$.
  - $\mathcal{H}$ $\sigma - \cap$-stable, if $(A_1, A_2, \ldots \in \mathcal{H} \Rightarrow \bigcap_{i=n}^\infty A_n \in \mathcal{H})$.
  - $\mathcal{H}$ $\cup$-stable, if $(A, B \in \mathcal{H} \Rightarrow A \cup B \in \mathcal{H})$.
  - $\mathcal{H}$ $\sigma - \cup$-stable, if $(A_1, A_2, \ldots \in \mathcal{H} \Rightarrow \bigcup_{i=n}^\infty A_n \in \mathcal{H})$.
  - $\mathcal{H}$ complement-stable, if $A \in \mathcal{H} \Rightarrow A^c \in \mathcal{H}$.
  - $\mathcal{H}$ set-difference-stable, if $(A, B \in \mathcal{H} \Rightarrow B \setminus A \in \mathcal{H})$.

Definition of some set-systems

- We write $A \uplus B$ for $A \cup B$ if $A \cap B = \emptyset$.
- Definition 1.1: Ω set, $\emptyset \neq \mathcal{H}, \mathcal{R}, \mathcal{F} \subseteq 2^\Omega$.
  - $\mathcal{H}$ is a *semi-ring*, if it is (i) $\cap$-stable and (ii) $\forall A, B \in \mathcal{H} \exists C_1, \dots, C_n \in \mathcal{H}$ with $B \setminus A = \biguplus_{i=1}^n C_i$.
  - $\mathcal{R}$ is a *ring*, if it is $\cup$-stable and set-difference-stable.
  - $\mathcal{F}$ is a σ*-field*, if $\Omega \in \mathcal{F}$, it is complement-stable and σ-$\cup$-stable. Then, $(\Omega, \mathcal{F})$ is called *measurable space*.

Connections between set-systems

	$\mathcal{C}$ semi-ring	$\mathcal{C}$ ring	$\mathcal{C}$ σ-field
$\mathcal{C}$ is $\cap$-stable	$\bullet$	$\circ$	$\circ$
$\mathcal{C}$ is σ-$\cap$-stable			$\circ$
$\mathcal{C}$ is $\cup$-stable		$\bullet$	$\circ$
$\mathcal{C}$ is σ-$\cup$-stable			$\bullet$
$\mathcal{C}$ is set-difference-stable		$\bullet$	$\circ$
$\mathcal{C}$ is complement-stable			$\bullet$
$B \setminus A = \biguplus_{i=1}^{n} C_i$	$\bullet$	$\circ$	$\circ$
$\Omega \in \mathcal{C}$			$\bullet$

Examples

- Semi-ring: Let $\Omega = \mathbb{R}$. Then,

$$\mathcal{H} := \{(a, b] : a, b \in \mathbb{Q}, a \leq b\} \text{ is a semi-ring.}$$

- σ-algebras: Trivial examples are $\{\emptyset, \Omega\}$ and 2^{Ω}.
If $\mathcal{F}'$ is a σ-field on Ω', and $f : \Omega \to \Omega'$. Then,

$$\sigma(f) := \{f^{-1}(A') : A' \in \mathcal{F}'\} \text{ is a } \sigma\text{-field on } \Omega.$$

Indeed: If $A', A'_1, A'_2, \ldots \in \sigma(f)$, then
$(f^{-1}(A'))^c = f^{-1}((A')^c) \in \sigma(f)$ and
$\bigcup_{n=1}^{\infty} f^{-1}(A'_n) = f^{-1}\left(\bigcup_{n=1}^{\infty} A'_n\right) \in \sigma(f)$.

Measure Theory for Probabilists
3. Generators and extensions

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Generated ring/σ-algebra

- Let $\mathcal{C} \subseteq 2^\Omega$. Then,

$$\mathcal{R}(\mathcal{C}) := \bigcap \left\{ \mathcal{R} \supseteq \mathcal{C} : \mathcal{R} \text{ ring} \right\},$$
$$\sigma(\mathcal{C}) := \bigcap \left\{ F \supseteq \mathcal{C} : \mathcal{F} \ \sigma\text{-field} \right\}$$

  are the ring and σ-algebra generated from $\mathcal{C}$,
- Example 1.6: Let $\mathcal{H} := \{[a, b), a \leq b, a, b \in \mathbb{Q}\}$. Then,

$$\mathcal{R}(\mathcal{H}) = \Big\{ \biguplus_{k=1}^{n} (a_k, b_k] : a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Q}, \\ a_k < b_k, k = 1, \ldots, n \text{ and } a_k < b_{k+1}, k = 1, \ldots, n-1 \Big\}$$

  is the ring generated from $\mathcal{H}$.

Generated ring

- Lemma 1.5: $\mathcal{H}$ semi-ring. Then,

$$\mathcal{R}(\mathcal{H}) = \left\{ \biguplus_{k=1}^{n} A_k : A_1, \ldots, A_n \in \mathcal{H} \text{ disjoint}, n \in \mathbb{N} \right\}$$

  is the ring generated from $\mathcal{H}$.

- Proof: $\mathcal{R}(\mathcal{H})$ is $\cap$-stable.
  To show: $\mathcal{R}(\mathcal{H})$ set-difference-stable. Let $A_1, \ldots, A_n \in \mathcal{H}$ and $B_1, \ldots, B_m \in \mathcal{H}$ be disjoint. Then,

$$\left(\biguplus_{i=1}^{n} A_i \right) \setminus \left(\biguplus_{j=1}^{m} B_j \right) = \biguplus_{i=1}^{n} \bigcap_{j=1}^{m} A_i \setminus B_j \in \mathcal{R}(\mathcal{H}).$$

  To show: $\mathcal{R}(\mathcal{H})$ is $\cup$-stable:

$$A \cup B = (A \cap B) \uplus (A \setminus B) \uplus (B \setminus A) \in \mathcal{R}(\mathcal{H})$$

Definitions from topology

- Ω some set. A set system $\mathcal{O} \subseteq 2^\Omega$ is called *topology* if (i) $\emptyset, \Omega \in \mathcal{O}$; (ii) if $\mathcal{O}$ is $\cap$-stable; (iii) if I is arbitrary and if $A_i \in \mathcal{O}, i \in I$, then $\bigcup_{i\in I} A_i \in \mathcal{O}$. The pair $(\Omega, \mathcal{O})$ is called *topological space*. Its members, i.e. every $A \in \mathcal{O}$, is called *open*; any set $A \subseteq \Omega$ with $A^c \in \mathcal{O}$ is called *closed*.
- (Ω, r) be a metric space and $B_\varepsilon(\omega) := \{\omega' \in \Omega : r(\omega, \omega') < \varepsilon\}$ an open ball and

$$\mathcal{B} := \{B_\varepsilon(\omega) : \varepsilon > 0, \omega \in \Omega\}. \tag{1}$$

Then,

$$\begin{aligned}\mathcal{O}(\mathcal{B}) &:= \{A \subseteq \Omega : \ \forall \omega \in A\, \exists B \in \mathcal{B} : \omega \in B \subseteq A\} \\ &= \Big\{ \bigcup_{B \in \mathcal{C}} B : \mathcal{C} \subseteq \mathcal{B} \Big\}\end{aligned}$$

is the topology generated by r.

Definitions from topology

- r is called *complete*, if every Cauchy-sequence converges.
- If there is some countable Ω' such that $\inf_{x' \in \Omega'} r(x, x') = 0$ for all $x \in \Omega$, we call (Ω, r) separable. In this case,

  $$\mathcal{B}' := \{B_r(\omega') : \omega' \in \Omega', r \in \mathbb{Q}_+\}$$

  is countable and $\mathcal{O}(\mathcal{B}') = \mathcal{O}(\mathcal{B})$.
- The space $(\Omega, \mathcal{O})$ is called Polish, if it is separable and completely metrizable.

Borel's σ-field

- Definition 1.7: $(\Omega, \mathcal{O})$ a topological space.

$$\mathcal{B}(\Omega) := \sigma(\mathcal{O})$$

  is the *Borel σ-algebra* on Ω. Sets in $\mathcal{B}(\Omega)$ are also called *(Borel-)measurable sets*.
- Lemma 1.8: Let $(\Omega, \mathcal{O})$ be a topological space with countable basis $\mathcal{C} \subseteq \mathcal{O}$. Then, $\sigma(\mathcal{O}) = \sigma(\mathcal{C})$.
- Proof: To show $\mathcal{O} \subseteq \sigma(\mathcal{C})$. Clear, since any $A \in \mathcal{O}$ can be represented as a countable union of sets from $\mathcal{C}$.

Borel σ-field generated by interavls

- Lemma 1.9: The set system

$$\mathcal{C}_1 = \{[-\infty, b] : b \in \mathbb{Q}\}$$

  generates $\mathcal{B}(\mathbb{R})$.
- Proof: Generate $(a, b]$ from $[-\infty, b] \setminus [-\infty, a]$, then $(a, b) = \bigcup_{i=1}^{\infty}(a, b - \frac{1}{n})$. These sets clearly generate $\mathcal{B}(\mathbb{R})$.

Measure Theory for Probabilists
4. Dynkin systems and compact systems

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January 2, 2024

Connections between set-systems

	$\mathcal{C}$ semi-ring	$\mathcal{C}$ ring	$\mathcal{C}$ σ-field
$\mathcal{C}$ is $\cap$-stable	$\bullet$	$\circ$	$\circ$
$\mathcal{C}$ is σ-$\cap$-stable			$\circ$
$\mathcal{C}$ is $\cup$-stable		$\bullet$	$\circ$
$\mathcal{C}$ is σ-$\cup$-stable			$\bullet$
$\mathcal{C}$ is set-difference-stable		$\bullet$	$\circ$
$\mathcal{C}$ is complement-stable			$\bullet$
$B \setminus A = \biguplus_{i=1}^n C_i$	$\bullet$	$\circ$	$\circ$
$\Omega \in \mathcal{C}$			$\bullet$

Dynkin systems

- Let $\mathcal{C} \subseteq 2^\Omega$. It is often easy to show that $\mathcal{C}$ is a (semi-)ring. However, it is hard to show that $\mathcal{C}$ is a σ-algebra. It is often easier to show that $\mathcal{C}$ is a Dynkin system:
- Definition 1.11: A set system $\mathcal{D}$ is called *Dynkin system* (on Ω) if (i) $\Omega \in \mathcal{D}$, (ii) it is set-difference-stable for subsets (i.e. $A, B \in \mathcal{D}$ and $A \subseteq B$ imply $B \setminus A \in \mathcal{D}$ and (iii) $A_1, A_2, \ldots \in \mathcal{D}$ and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ imply $\bigcup_{n=1}^\infty A_n \in \mathcal{D}$.
- Goal is Theorem 1.13:
  A $\cap$-stable Dynkin system is a σ-algebra.
- Example 1.12:
  $\mathcal{F}$ σ-algebra $\Rightarrow$ $\mathcal{F}$ Dynkin-system
  $\mathcal{F}$ Dynkin system $\Rightarrow$ $\mathcal{F}$ complement-stable

Theorem 1.13:

- $\mathcal{D}$ Dynkin system, $\mathcal{C} \subseteq \mathcal{D}$ is $\cap$-stable $\Rightarrow \sigma(\mathcal{C}) \subseteq \mathcal{D}$.
- Proof: Set

$$\lambda(\mathcal{C}) := \bigcap\{\mathcal{D}' \supseteq \mathcal{C}, \mathcal{D}' \text{ Dynkin-system}\} \supseteq \lambda(\mathcal{C}).$$

Claim: $\lambda(\mathcal{C})$ is a σ-algebra ($\Rightarrow \sigma(\mathcal{C}) \subseteq \sigma(\lambda(\mathcal{C})) = \lambda(\mathcal{C}) \subseteq \mathcal{D}$)
Suffices: $\lambda(\mathcal{C})$ is $\cap$-stable.
Then, $A \cup B = (A^c \cap B^c)^c$, so $\lambda(\mathcal{C})$ is $\cup$-stable and for $A_1, A_2, ... \in \lambda(\mathcal{C})$, we find $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n} A_i \in \lambda(\mathcal{C})$.
For $B \in \mathcal{C}$, set

$$\mathcal{D}_B := \{A \subseteq \Omega : A \cap B \in \lambda(\mathcal{C})\} \supseteq \mathcal{C}.$$

Then $\mathcal{D}_B$ is a Dynkin system...
So, $\lambda(\mathcal{C}) \subseteq \mathcal{D}_B$. So, for an $A \in \lambda(\mathcal{C})$,

$$\mathcal{B}_A := \{B \subseteq \Omega : A \cap B \in \lambda(\mathcal{C})\} \supseteq \lambda(\mathcal{C}) \text{ is Dynkin system.}$$

Compact sets

- $J \subseteq_f I$ if $J \subseteq I$ and J is finite
- Definition A.7: (Ω, r) metric space, $K \subseteq \Omega$.
  1. K is *compact* if every open cover has a finite partial cover: If $O_i \in \mathcal{O}, i \in I$ and $K \subseteq \bigcup_{i \in I} O_i$, then there is $J \subset_f I$ with $K \subseteq \bigcup_{i \in J} O_i$.
  2. K is *relatively compact* if $\overline{K}$ is compact.
  3. K is *relatively sequentially compact* if for every sequence in K there is a convergent subsequence.
  4. $K \subseteq \Omega$ is *totally bounded* if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ and $\omega_1, \ldots, \omega_N \in K$ such that $K \subseteq \bigcup_{n=1}^{N} B_\varepsilon(\omega_n)$.
- Lemma A.8:: $K \subseteq \Omega$ compact $\Rightarrow$ K is closed.

Compact sets

- Proposition A.9: $K \subseteq \Omega$.
  1. K is relatively compact.
  2. If $F_i \subseteq \overline{K}$ is closed, $i \in I$, and $\bigcap_{i \in I} F_i = \emptyset$, then there is $J \subseteq_f I$ with $\bigcap_{i \in J} F_i = \emptyset$.
  3. K is relatively sequentially compact.
  4. K is totally bounded.

  Then

  $$4. \Longleftarrow 1. \iff 2. \Longrightarrow 3.$$

  Furthermore, $3. \Longrightarrow 2.$ also holds if $(\Omega, \mathcal{O})$ is separable and $4. \Longrightarrow 3.$ if (Ω, r) is complete.

Compact systems

- Definition 1.14: $\mathcal{K}$ $\cap$-stable is *compact system* if $\bigcap_{n=1}^\infty K_n = \emptyset$ with $K_1, K_2, \ldots \in \mathcal{K}$ implies that there is a $N \in \mathbb{N}$ with $\bigcap_{n=1}^N K_n = \emptyset$.
- Example 1.15: $\mathcal{K} \subseteq \{K \subseteq \Omega : K \text{ compact}\}$ $\cap$-stable is compact system.
  Indeed: Let $\bigcap_{n=1}^\infty K_n = \emptyset$. Then, K_1 and $L_n := K_1 \cap K_n \subseteq K_1$ are compact and (because of the compactness of K_1) there is an N with $\bigcap_{n=1}^N K_n = \emptyset$ due to Proposition A.9.

Compact systems

- Lemma 1.16: $\mathcal{K}$ compact system. Then,

$$\mathcal{K}_\cup := \left\{ \bigcup_{i=1}^n K_i : K_1, \dots, K_n \in \mathcal{K}, n \in \mathbb{N} \right\}$$

  is also a compact system.

- Proof: $\mathcal{K}_\cup$ is $\cap$-stable. Let $L_1 = \bigcup_{j=1}^{m_1} K_j^1, L_2 = \bigcup_{j=1}^{m_2} K_j^2, \dots \in \mathcal{K}_\cup$ with $\bigcap_{n=1}^N L_n \neq \emptyset$ for all $N \in \mathbb{N}$. To show: $\bigcap_{n=1}^\infty L_n \neq \emptyset$. Use induction over N for:

  For every $N \in \mathbb{N}$ there are sets $K_1, \dots, K_N \in \mathcal{K}$ with $K_n \subseteq L_n$, $n = 1, \dots, N$, such that for all $k \in \mathbb{N}_0$ we have $K_1 \cap \dots \cap K_N \cap L_{N+1} \cap \dots \cap L_{N+k} \neq \emptyset$.

  Then, use $k = 0$. So we see that there are $K_1, K_2, \dots \in \mathcal{K}$ and $K_n \subseteq L_n$, $n \in \mathbb{N}$ with $\bigcap_{n=1}^N K_n \neq \emptyset$ for all $N \in \mathbb{N}$. Hence, $\emptyset \neq \bigcap_{n=1}^\infty K_n \subseteq \bigcap_{n=1}^\infty L_n$.

Measure Theory for Probabilists
5. Set functions and outer measures

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Definition 2.1

- For $\mathcal{F} \subseteq 2^{\Omega}$, we call $\mu : \mathcal{F} \to \overline{\mathbb{R}}_+$ a set function.
- μ is *finitely additive* if

$$\mu\left(\biguplus_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k).$$

  for disjoint $A_1, \ldots, A_n \in \mathcal{F}$.
- $\mu : \mathcal{F} \to \mathbb{R}_+$ is *σ-additive* if the same holds for $n = \infty$.
- If $\mathcal{F}$ is a σ-algebra, and μ is σ-additive, μ is a *measure* and $(\Omega, \mathcal{F}, \mu)$ is a *measure space*.
- If $\mu(\Omega) < \infty$, then μ is a *finite measure*; if $\mu(\Omega) = 1$, μ is a *probability measure*. Then, $(\Omega, \mathcal{F}, \mu)$ is a *probability space*.

Definition 2.1

- μ is called *sub-additive* if

$$\mu\Big(\bigcup_{k=1}^{n} A_k\Big) \leq \sum_{k=1}^{n} \mu(A_k).$$

  for any $A_1, \ldots, A_n \in \mathcal{F}$.
- $\mu : \mathcal{F} \to \mathbb{R}_+$ is *σ-sub-additive* if the same holds for $n = \infty$.
- μ is *monotone* if $(A \subseteq B \Rightarrow \mu(A) \leq \mu(B))$
- A σ-subadditive, monotone $\mu^* : 2^\Omega \to \mathbb{R}_+$ with $\mu^*(\emptyset) = 0$ is an *outer measure*.
- A set $A \subseteq \Omega$ is called *μ^*-measurable* if

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c), \qquad E \subseteq \Omega.$$

Definition 2.1

- If there is $\Omega_1, \Omega_2, \ldots \in \mathcal{F}$ with $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$ and $\mu(\Omega_n) < \infty$ for all $n = 1, 2, \ldots$, then μ is *σ-finite*.
- $\mathcal{F}$ $\cap$-stable. μ is inner $\mathcal{K}-$regular if for all $A \in \mathcal{F}$

$$\mu(A) = \sup_{\mathcal{K} \ni K \subseteq A} \mu(K).$$

- $(\Omega, \mathcal{O})$ topological space, μ measure on $\mathcal{B}(\mathcal{O})$. The smallest closed set F with $\mu(F^c) = 0$ is called the *support of* μ.

Examples

- Let $\mathcal{H} = \{(a,b] : a,b \in \mathbb{Q}, a \le b\}$. Then, $\mu((a,b]) := b - a$ defines an additive, σ-finite set function.
- Let $\omega' \in \Omega$. Then, $\delta_{\omega'}(A) := 1_{\{\omega' \in A\}}$ is a probability measure.
- $\mu := \sum_{i \in I} \delta_{\omega_i}$ is a *counting measure*.
- $\mu_i, i \in I$ measures and $a_i \in \mathbb{R}_+, i \in I$. Then, $\sum_{i \in I} a_i \mu_i$ is also a measure, e.g. the Poisson distribution on $2^{\mathbb{N}_0}$,

$$\mu_{\mathrm{Poi}(\gamma)} := \sum_{k=0}^{\infty} e^{-\gamma} \frac{\gamma^k}{k!} \cdot \delta_k,$$

the geometric distribution

$$\mu_{\mathrm{geo}(p)} := \sum_{k=1}^{\infty} (1-p)^{k-1} p \cdot \delta_k,$$

the binomial distribution

$$\mu_{B(n,p)} := \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot \delta_k.$$

Unions and disjoint unions

- Lemma 2.4: $\mathcal{H}$ semi-ring, $A, A_1, ..., A_n \in \mathcal{H}$. Then, there are $B_1, ..., B_m \in \mathcal{H}$ pairwise disjoint and $A \setminus \bigcup_{i=1}^n A_i = \biguplus_{j=1}^m B_j$.
- Proof: Induction on n. If $n = 1$, clear. Assume the assertion holds for some n, i.e. there is $B_1, ..., B_m$ with $A \setminus \bigcup_{i=1}^n A_i = \biguplus_{j=1}^m B_j$. Then, write $B_j \setminus A_{n+1} = \biguplus_{k=1}^{k_j} C_k^j$ for $C_1^j, ..., C_{k_j}^j \in \mathcal{H}$. Then,

$$A \setminus \bigcup_{i=1}^{n+1} A_i = \Big(A \setminus \bigcup_{i=1}^{n} A_i\Big) \setminus A_{n+1} = \biguplus_{j=1}^{m} B_j \setminus A_{n+1} = \biguplus_{j=1}^{m} \biguplus_{k=1}^{k_j} C_k^j.$$

Set-functions on semi-rings

- Lemma 2.5: $\mathcal{H}$ semi-ring, $\mu : \mathcal{H} \to [0, \infty]$ additive. Then, m is monotone and sub-additive.
- Proof: Monotonicity for $A, B \in \mathcal{H}$ with $A \subseteq B$ and $C_1, ..., C_k \in \mathcal{H}$ with $B \setminus A = \biguplus_{i=1}^k C_i$. Write $\mu(A) \leq \mu(A) + \sum_{i=1}^k \mu(C_i) = \mu(B)$.
  Claim: $\biguplus_{I \in \mathcal{I}} A_i \subseteq A \Rightarrow \sum_{i=1}^n \mu(A_i) \leq m(A)$.
  Write $A \setminus \biguplus_{i=1}^n A_i = \biguplus_{j=1}^m B_j$. Then,

$$\mu(A) = \mu\Big(\biguplus_{i=1}^n A_i \uplus \biguplus_{j=1}^m B_j\Big) = \sum_{i=1}^n \mu(A_i) + \sum_{j=1}^m \mu(B_j) \geq \sum_{i=1}^n \mu(A_i).$$

  Sub-additivity: To show $\mu\Big(\bigcup_{i=1}^n A_i\Big) \leq \sum_{i=1}^n \mu(A_i)$. Write

$$\mu\Big(\bigcup_{i=1}^n A_i\Big) = \mu\Big(\biguplus_{i=1}^n \Big(A_i \setminus \bigcup_{j=1}^{i-1} A_j\Big)\Big) = \sum_{k=1}^n \sum_{k=1}^{k_i} \mu(C_k^i) \leq \sum_{i=1}^n \mu(A_i).$$

Set-functions on semi-rings

- Lemma 2.5: μ is σ-additive iff μ is σ-sub-additive.
- Proof: '$\Rightarrow$': Copy the proof of sub-additivity using $n = \infty$.
  '$\Leftarrow$': Let $A = \biguplus_{i=1}^{\infty} A_i \in \mathcal{H}$.
  Then, $\sum_{i=1}^{n} \mu(A_i) \leq \mu(A)$ by monotonicity and

$$\sum_{i=1}^{\infty} \mu(A_i) = \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} \mu(A_i) \leq \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

  by σ-sub-additivity.

Extension of set-functions on semi-rings

- Lemma 2.6: $\mathcal{H}$ semi-ring, $\mathcal{R}$ ring generated by $\mathcal{H}$, μ additive on $\mathcal{H}$. Then,

$$\widetilde{\mu}\Big(\biguplus_{i=1}^{n} A_i\Big) := \sum_{i=1}^{n} \mu(A_i)$$

  $\widetilde{\mu}$ is the only additive extension of μ on $\mathcal{R}$ that coincides with μ on $\mathcal{H}$.
- Proof: Suffices to show that $\widetilde{\mu}$ is well-defined. Let $\biguplus_{i=1}^{m} A_i = \biguplus_{j=1}^{n} B_j$. Since

$$A_i = \biguplus_{j=1}^{n} A_i \cap B_j, \qquad B_j = \biguplus_{i=1}^{m} A_i \cap B_j,$$

$$\sum_{i=1}^{m} \mu(A_i) = \sum_{i=1}^{m}\sum_{j=1}^{n} \mu(A_i \cap B_j) = \sum_{j=1}^{n}\sum_{i=1}^{m} \mu(A_i \cap B_j) = \sum_{j=1}^{n} \mu(B_j).$$

Inclusion exclusion principle

- Proposition 2.7: μ be additive set function on ring $\mathcal{R}$ and I finite. Then for $A_i \in \mathcal{R}$, $i \in I$, it holds that

$$\mu\Big(\bigcup_{i\in I} A_i\Big) = \sum_{J\subseteq I} (-1)^{|J|+1} \mu\Big(\bigcap_{j\in J} A_j\Big)$$

  In particular, if $I = \{1, 2\}$,

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cap A_2).$$

- Proof for $|I| = 2$: $A_1 \cup A_2 = A_1 \uplus (A_2 \setminus A_1)$ and $(A_2 \setminus A_1) \uplus (A_1 \cap A_2) = A_2$.

Measure Theory for Probabilists
6. σ-additivity

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Proposition 2.8

- μ is σ-additive iff
$$\mu\Big(\biguplus_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$
- μ is σ-sub-additive iff
$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} \mu(A_n).$$
- μ is continuous from below, if for $A, A_1, A_2, \ldots$ and $A_1 \subseteq A_2 \subseteq \ldots$ with $A = \bigcup_{n=1}^{\infty} A_n$,
$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$
- μ is continuous from above (in the $\emptyset$), if for $A(=\emptyset), A_1, A_2, \ldots,\ \mu(A_1) < \infty$ and $A_1 \supseteq A_2 \supseteq \ldots$ with $A = \bigcap_{n=1}^{\infty} A_n$,
$$(0 =)\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

Proposition 2.8

- Let $\mathcal{R}$ be a ring and $\mu : \mathcal{R} \to \overline{\mathbb{R}}_+$ be additive and $\mu(A) < \infty$ for all $A \in \mathcal{R}$. Then, the following are equivalent:
  1. μ is σ-additive;
  2. μ is σ-subadditive;
  3. μ is continuous from below;
  4. μ is continuous from above in $\emptyset$;
  5. μ is continuous from above.
- Proof: 1.$\Leftrightarrow$2., 5.$\Rightarrow$4.: clear.
  1.$\Rightarrow$3.: With $A_0 = \emptyset$, $A = \biguplus_{n=1}^{\infty} A_n \setminus A_{n-1}$
  3.$\Rightarrow$1.: Set $A_N = \biguplus_{n=1}^{N} B_n$,
  4.$\Rightarrow$5.: With $B_n := A_n \setminus A \downarrow \emptyset$,
  $\mu(A_n) = \mu(B_n) + \mu(A) \xrightarrow{n \to \infty} \mu(A)$.
  3.$\Rightarrow$4.: Set $B_n := A_1 \setminus A_n \uparrow A_1$. Then,
  $\mu(A_1) = \lim_{n\to\infty} \mu(B_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$.
  4.$\Rightarrow$3. Set $B_n := A \setminus A_n \downarrow \emptyset$. Then,
  $0 = \lim_{n\to\infty} \mu(B_n) = \mu(A) - \lim_{n\to\infty} \mu(A_n)$.

Inner regularity of measures on Polish spaces

- Lemma 2.9: $(\Omega, \mathcal{O})$ Polish, μ finite, $\varepsilon > 0$. There exists $K \subseteq \Omega$ compact with $\mu(\Omega \setminus K) < \varepsilon$.
- Proof: There is $\{\omega_1, \omega_2, \dots\} \subseteq \Omega$ dense, so $\Omega = \bigcup_{k=1}^{\infty} B_{1/n}(\omega_k)$. μ is continuous from above $\Rightarrow$

$$0 = \mu\Big(\Omega \setminus \bigcup_{k=1}^{\infty} B_{1/n}(\omega_k)\Big) = \lim_{N\to\infty} \mu\Big(\Omega \setminus \bigcup_{k=1}^{N} B_{1/n}(\omega_k)\Big).$$

Take $N_n \in \mathbb{N}$ with $\mu\Big(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k^n)\Big) < \varepsilon/2^n$ and $A := \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)$ totally bounded, hence relatively compact with

$$\mu(\Omega \setminus \overline{A}) \leq \mu(\Omega \setminus A) \leq \mu\Big(\bigcup_{n=1}^{\infty} \Big(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)\Big)\Big)$$

$$\leq \sum_{n=1}^{\infty} \mu\Big(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)\Big) < \varepsilon.$$

Inner regularity and σ-additivity

- Theorem 2.10: $\mathcal{H}$ semi-ring, $\mu : \mathcal{H} \to \mathbb{R}_+$ finite, finitely additive and inner $\mathcal{K} \subseteq \mathcal{H}$-regular. Then μ is σ-additive.
- Proof:Wlog, $\mathcal{H}$ is ring and $\mathcal{K} = \mathcal{K}_\cup$
  To show: μ is continuous from above in $\emptyset$. Let $A_1, A_2, \dots \in \mathcal{H}$ with $A_1 \supseteq A_2 \supseteq \cdots$ and $\bigcap_{n=1}^\infty A_n = \emptyset$ and $\varepsilon > 0$.
  Choose $K_1, K_2, \dots \in \mathcal{K}$ with $K_n \subseteq A_n, n \in \mathbb{N}$ and

$$\mu(A_n) \leq \mu(K_n) + \varepsilon 2^{-n}.$$

  Then, $\bigcap_{n=1}^\infty K_n \subseteq \bigcap_{n=1}^\infty A_n = \emptyset$, so there is $N \in \mathbb{N}$ with $\bigcap_{n=1}^N K_n = \emptyset$. From this,

$$A_N = A_N \cap \left(\bigcup_{n=1}^N K_n^c \right) = \bigcup_{n=1}^N A_N \setminus K_n \subseteq \bigcup_{n=1}^N A_n \setminus K_n.$$

  By subadditivity and monotonicity of μ, for $m \geq N$,

$$\mu(A_m) \leq \mu(A_N) \leq \sum_{n=1}^N \mu(A_n \setminus K_n) \leq \varepsilon \sum_{n=1}^N 2^{-n} \leq \varepsilon.$$

Measure Theory for Probabilists
7. Uniqueness and extension of set functions

Peter Pfaffelhuber

January 8, 2024

Question

- When does an additive set-function μ on $\mathcal{H}$ uniquely extend to a measure $\widetilde{\mathcal{H}}$ on $\sigma(\mathcal{H})$?
- Uniqueness: Proposition 2.11: Let $\mathcal{C} \subseteq 2^{\Omega}$ be $\cap$-stable, and μ, ν be σ-finite measures on $\sigma(\mathcal{C})$. Then,

$$\mu = \nu \qquad \iff \qquad \mu|_{\mathcal{C}} = \nu|_{\mathcal{C}}.$$

- Existence: See Carathéodory's Extension Theorem 2.13: Let μ^* be an outer measure. Then, $\mathcal{F}^*$ the set of μ^*-measurable sets is a σ-algebra and $\mu := \mu^*|_{\mathcal{F}^*}$ is a measure.

Theorem 2.16

	Lemma 2.5	Theorem 2.10	Theorem 2.16
μ additive	$\circ$	$\circ$	
μ finite		$\circ$	
μ σ-finite			$\circ$
μ defined on semi-ring	$\circ$	$\circ$	$\circ$
hline μ σ-additive	$\circ/\bullet$	$\bullet$	$\circ$
hline μ σ-subadditive	$\bullet/\circ$		
μ inner $\mathcal{K}$-regular		$\circ$	
μ extends uniquely to $\sigma(\mathcal{H})$			$\bullet$

Proposition 2.11

- Let $\mathcal{C} \subseteq 2^{\Omega}$ be $\cap$-stable, and μ, ν be σ-finite measures on $\sigma(\mathcal{C})$. Then,

$$\mu = \nu \qquad \Longleftrightarrow \qquad \mu|_{\mathcal{C}} = \nu|_{\mathcal{C}}.$$

- Proof for finite μ, ν with $\mu(\Omega) = \nu(\Omega)$: $\Rightarrow$: clear
  $\Leftarrow$: Let

$$\mathcal{D} := \{B \in \mathcal{F} : \mu(A) = \nu(A)\} \supseteq \mathcal{H}.$$

  To show: $\mathcal{D}$ is Dynkin. $\Rightarrow \sigma(\mathcal{H}) \subseteq \mathcal{D}$ by Theorem 1.13.
  - $B, C \in \mathcal{D}, B \subseteq C \Rightarrow \mu(C \setminus B) = \mu(C) - \mu(B) = \nu(C) - \nu(B) = \nu(C \setminus B)$, i.e. $C \setminus B \in \mathcal{D}$.
  - $B_1, B_2, \dots \in \mathcal{D}$ with $B_1 \subseteq B_2 \subseteq B_3 \subseteq \dots \in \mathcal{D}$ and $B = \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$, then from continuity from below,

$$\mu(B) = \lim_{n \to \infty} \mu(B_n) = \lim_{n \to \infty} \nu(B_n) = \nu(B) \quad \Rightarrow B \in \mathcal{D}.$$

Theorem 2.13

- A σ-subadditive, monotone $\mu^* : 2^\Omega \to \mathbb{R}_+$ with $\mu^*(\emptyset) = 0$ is an *outer measure*.
- A set $A \subseteq \Omega$ is called μ^**-measurable* if

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c), \qquad E \subseteq \Omega.$$

- Theorem 2.13: Let μ^* be an outer measure. Then, $\mathcal{F}^*$ the set of μ^*-measurable sets is a σ-algebra and $\mu := \mu^*|_{\mathcal{F}^*}$ is a measure. Furthermore, $\mathcal{N} := \{N \subseteq \Omega : \mu^*(N) = 0\} \subseteq \mathcal{F}^*$.

Theorem 2.13

- Let μ^* be an outer measure. Then, $\mathcal{F}^*$ the set of μ^*-measurable sets is a σ-algebra and $\mu := \mu^*|_{\mathcal{F}^*}$ is a measure.
- Proof: Show:
  - $\emptyset \in \mathcal{F}^*$, since $\mu^*(E) = \mu^*(E \cap \emptyset) + \mu^*(E \cap \Omega)$.
  - $A \in \mathcal{F}^* \Rightarrow A^c \in \mathcal{F}^*$
  - $A, B \in \mathcal{F}^* \Rightarrow A \cap B \in \mathcal{F}^*$, since

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c)\\ &= \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) \cap B^c) + \mu^*(E \cap A^c)\\ &\geq \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \cap B)^c) \geq \mu^*(E),\end{aligned}$$

  - $A_1, A_2, \dots \in \mathcal{F}^*$ disjoint, $B_n = \biguplus_{k=1}^n A_k \in \mathcal{F}^*$, $B_n \uparrow B$. Show $\mu^*(E \cap B_n) = \sum_{k=1}^n \mu^*(E \cap A_k)$ by induction on n:

$$\begin{aligned}\mu^*(E \cap B_{n+1}) &= \mu^*(E \cap B_{n+1} \cap B_n) + \mu^*(E \cap B_{n+1} \cap B_n^c)\\ &= \mu^*(E \cap B_n) + \mu^*(E \cap A_{n+1}) = \sum_{k=1}^{n+1} \mu^*(E \cap A_k).\end{aligned}$$

Theorem 2.13

- Let μ^* be an outer measure. Then, $\mathcal{F}^*$ the set of μ^*-measurable sets is a σ-algebra and $\mu := \mu^*|_{\mathcal{F}^*}$ is a measure.
- Then, $\mu^*(E \cap B) = \sum_{k=1}^{\infty} \mu^*(E \cap A_k) = \lim_{n\to\infty} \mu^*(E \cap B_n)$ since

$$\mu^*(E \cap B) \leq \sum_{k=1}^{\infty} \mu^*(E \cap A_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu^*(E \cap A_k)$$
$$= \lim_{n\to\infty} \mu^*(E \cap B_n) \leq \mu^*(E \cap B),$$

- $B \in \mathcal{F}^*$, since $B_1, B_2, ... \in \mathcal{F}^*$, so

$$\mu^*(E) = \lim_{n\to\infty} \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c)$$
$$\geq \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E).$$

- So, $\mathcal{F}^*$ is a σ-algebra and μ^* is σ-additive on $\mathcal{F}^*$, i.e. $\mu = \mu^*|_{\mathcal{F}^*}$ is a measure.

Theorem 2.13

- $\mathcal{N} := \{N \subseteq \Omega : \mu^*(N) = 0\} \subseteq \mathcal{F}^*$.
- $N \in \mathcal{N}$ are called *(μ^*-)null sets*.
  If $A^c \in \mathcal{N}$, we say that A holds (μ)-*almost everywhere* or *almost surely*.
- Proof: For $N \in \mathcal{N}$, by monotonicity $\mu^*(E \cap N) = 0$, so

$$\begin{aligned}\mu^*(E \cap N^c) + \mu^*(E \cap N) \geq \mu^*(E) &\geq \mu^*(E \cap N^c) \\ &= \mu^*(E \cap N^c) + \mu^*(E \cap N).\end{aligned}$$

Zweite Folie

- Test

Zweite Folie

- Test

Proposition 2.8

- μ is σ-additive iff

$$\mu\Big(\biguplus_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

- μ is σ-sub-additive iff

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

- μ is continuous from below, if for $A, A_1, A_2, \ldots$ and $A_1 \subseteq A_2 \subseteq \ldots$ with $A = \bigcup_{n=1}^{\infty} A_n$,

$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

- μ is continuous from above (in the $\emptyset$), if for $A(=\emptyset), A_1, A_2, \ldots,$ $\mu(A_1) < \infty$ and $A_1 \supseteq A_2 \supseteq \ldots$ with $A = \bigcap_{n=1}^{\infty} A_n$,

$$(0 =)\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

Proposition 2.8

- Let $\mathcal{R}$ be a ring and $\mu : \mathcal{R} \to \overline{\mathbb{R}}_+$ be additive and $\mu(A) < \infty$ for all $A \in \mathcal{R}$. Then, the following are equivalent:
  1. μ is σ-additive;
  2. μ is σ-subadditive;
  3. μ is continuous from below;
  4. μ is continuous from above in $\emptyset$;
  5. μ is continuous from above.
- Proof: 1.⇔2., 5.⇒4.: clear.
  1.⇒3.: With $A_0 = \emptyset$, $A = \biguplus_{n=1}^{\infty} A_n \setminus A_{n-1}$
  3.⇒1.: Set $A_N = \biguplus_{n=1}^{N} B_n$,
  4.⇒5.: With $B_n := A_n \setminus A \downarrow \emptyset$,
  $\mu(A_n) = \mu(B_n) + \mu(A) \xrightarrow{n\to\infty} \mu(A)$.
  3.⇒4.: Set $B_n := A_1 \setminus A_n \uparrow A_1$. Then,
  $\mu(A_1) = \lim_{n\to\infty} \mu(B_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$.
  4.⇒3. Set $B_n := A \setminus A_n \downarrow \emptyset$. Then,
  $0 = \lim_{n\to\infty} \mu(B_n) = \mu(A) - \lim_{n\to\infty} \mu(A_n)$.

Inner regularity of measures on Polish spaces

- Lemma 2.9: $(\Omega, \mathcal{O})$ Polish, μ finite, $\varepsilon > 0$.
  There exists $K \subseteq \Omega$ compact with $\mu(\Omega \setminus K) < \varepsilon$.
- Proof: There is $\{\omega_1, \omega_2, \dots\} \subseteq \Omega$ dense, so $\Omega = \bigcup_{k=1}^{\infty} B_{1/n}(\omega_k)$. μ is continuous from above $\Rightarrow$

$$0 = \mu\Big(\Omega \setminus \bigcup_{k=1}^{\infty} B_{1/n}(\omega_k)\Big) = \lim_{N\to\infty} \mu\Big(\Omega \setminus \bigcup_{k=1}^{N} B_{1/n}(\omega_k)\Big).$$

  Take $N_n \in \mathbb{N}$ with $\mu\Big(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k^n)\Big) < \varepsilon/2^n$ and $A := \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)$ totally bounded, hence relatively compact with

$$\mu(\Omega \setminus \overline{A}) \le \mu(\Omega \setminus A) \le \mu\Big(\bigcup_{n=1}^{\infty} \Big(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)\Big)\Big)$$

$$\le \sum_{n=1}^{\infty} \mu\Big(\Omega \setminus \bigcup_{k=1}^{N_n} B_{1/n}(\omega_k)\Big) < \varepsilon.$$

Inner regularity and σ-additivity

- Theorem 2.10: $\mathcal{H}$ semi-ring, $\mu : \mathcal{H} \to \mathbb{R}_+$ finite, finitely additive and inner $\mathcal{K} \subseteq \mathcal{H}$-regular. Then μ is σ-additive.
- Proof:Wlog, $\mathcal{H}$ is ring and $\mathcal{K} = \mathcal{K}_\cup$
  To show: μ is continuous from above in $\emptyset$. Let $A_1, A_2, \dots \in \mathcal{H}$ with $A_1 \supseteq A_2 \supseteq \cdots$ and $\bigcap_{n=1}^\infty A_n = \emptyset$ and $\varepsilon > 0$.
  Choose $K_1, K_2, \dots \in \mathcal{K}$ with $K_n \subseteq A_n, n \in \mathbb{N}$ and

$$\mu(A_n) \leq \mu(K_n) + \varepsilon 2^{-n}.$$

  Then, $\bigcap_{n=1}^\infty K_n \subseteq \bigcap_{n=1}^\infty A_n = \emptyset$, so there is $N \in \mathbb{N}$ with $\bigcap_{n=1}^N K_n = \emptyset$. From this,

$$A_N = A_N \cap \left(\bigcup_{n=1}^N K_n^c\right) = \bigcup_{n=1}^N A_N \setminus K_n \subseteq \bigcup_{n=1}^N A_n \setminus K_n.$$

  By subadditivity and monotonicity of μ, for $m \geq N$,

$$\mu(A_m) \leq \mu(A_N) \leq \sum_{n=1}^N \mu(A_n \setminus K_n) \leq \varepsilon \sum_{n=1}^N 2^{-n} \leq \varepsilon.$$

Measure Theory for Probabilists
8. Measures on $\mathbb{R}$ and image measures

Peter Pfaffelhuber

January 9, 2024

Lebesgue measure

- Proposition 2.18: There is exactly one measure λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with

$$\lambda((a,b]) = b - a$$

for $a, b \in \mathbb{Q}$ with $a \leq b$.
- Proof: $\mathcal{H} = \{(a,b] : a, b \in \mathbb{Q}, a \leq b\}$ is a semi-ring with $\sigma(\mathcal{H}) = \mathcal{B}(\mathbb{R})$.
  σ-additivity: let $a_1, a_2, \ldots$ be such that $\bigcup_{n=1}^{\infty}(a_{n+1}, a_n] = (a,b] \in \mathcal{H}$, i.e., $b = a_1$ and $a_n \downarrow a$. Then,

$$\lambda(a,b] = b - a = a_1 - \lim_{N\to\infty} a_N = \sum_{n=1}^{\infty} a_n - a_{n+1} = \sum_{n=1}^{\infty} \lambda((a_{n+1}, a_n]).$$

Conclude with Theorem 2.16.

σ-finite measures on $\mathbb{R}$

- Proposition 2.19: $\mu : \mathcal{B}(\mathbb{R}) \to \overline{\mathbb{R}}_+$ is a σ-finite measure iff there is $G : \mathbb{R} \to \mathbb{R}$, non-decreasing and right-continuous with

$$\mu((a,b]) = G(b) - G(a), \qquad a, b \in \mathbb{Q}, a \leq b. \qquad (*)$$

  If $\widetilde{G}$ also satisfies $(*)$, then $\widetilde{G} = G + c$ for some $c \in \mathbb{R}$.
- Proof: '$\Rightarrow$': Define $G(0) = 0$ and
  $$G(x) := \begin{cases} \mu((0,x]), & x > 0, \\ -\mu((x,0]), & x < 0. \end{cases}$$
  '$\Leftarrow$': Similar to the proof of Proposition 2.18.
  Let $\widetilde{G}$ satisfy $(*)$. Then, for $a \in \mathbb{R}$,

$$\widetilde{G}(b) = \widetilde{G}(a) + \mu((a,b]) = G(b) + \widetilde{G}(a) - G(a),$$

  and the assertion follows with $c = \widetilde{G}(a) - G(a)$.

Probability measures on $\mathbb{R}$

- Corollary 2.20: $\mu : \mathcal{B}(\mathbb{R}) \to [0,1]$ is probability measure iff there is $F : \mathbb{R} \to [0,1]$ non-decreasing and right-continuous with $\lim_{b\to\infty} F(b) = 1$ and

$$\mu((a,b]) = F(b) - F(a), \qquad a, b \in \mathbb{Q}, a \le b.$$

  F is uniquely defined by μ.
  F is called the distribution function of μ.

Examples

- Let $f : \mathbb{R} \to \mathbb{R}_+$ be a density (piecewise continuous with $\int_{-\infty}^{\infty} f(x)dx = 1$). A densitiy defines a distribution function via
$$F(x) := \int_{-\infty}^{x} f(a)da,$$
therefore uniquely a probability measures.

$$F_{U(0,1)}(x) = \int_{-\infty}^{x} 1_{[0,1]}(a)da = \begin{cases} 0, & x \leq 0, \\ x, & 0 < x \leq 1, \\ 1, & x > 1, \end{cases}$$

$$F_{\exp(\lambda)}(x) = \int_{-\infty}^{x} 1_{[0,\infty)}(a)\lambda e^{-\lambda a}da = 1 - e^{-\lambda x}$$

$$F_{N(\mu,\sigma^2)}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} \exp\Big(- \frac{(a-\mu)^2}{2\sigma^2}\Big) da =: \Phi(x)$$

Image measures

- If $\mathcal{F}'$ is a σ-field on Ω', and $f : \Omega \to \Omega'$. Then,
$$\sigma(f) := \{f^{-1}(A') : A' \in \mathcal{F}'\} \text{ is a } \sigma\text{-field on } \Omega.$$
- Definition 2.23: $(\Omega, \mathcal{F}, \mu)$ measure space, $(\Omega', \mathcal{F}')$ measurable space, $f : \Omega \to \Omega'$ with $\sigma(f) \subseteq \mathcal{F}$. Then,
$$\mathcal{F}' \ni A' \mapsto f_*\mu(A') := \mu(f^{-1}(A')) = \mu(f \in A')$$
is the *image measure* of f under μ.
If $\mathbb{P}$ is a probability measure, we call $X_*\mu$ the distribution of X under $\mathbb{P}$.
- Proposition 2.25: $f_*\mu$ is a measure on $\mathcal{F}'$.
- Proof: $A'_1, A'_2, \dots \in \mathcal{F}'$ disjoint, then
$$\begin{aligned} f_*\mu\Big(\biguplus_{n=1}^\infty A'_n\Big) &= \mu\Big(f^{-1}\Big(\biguplus_{n=1}^\infty A'_n\Big)\Big) \\ &= \mu\Big(\biguplus_{n=1}^\infty (f^{-1}(A'_n)\Big) = \sum_{n=1}^\infty \mu(f^{-1}(A'_n)) = \sum_{n=1}^\infty f_*\mu(A'_n). \end{aligned}$$

Examples

- For $\Omega = [0,1]$, $\mathcal{H} := \{[0,b) : 0 \le b \le 1\}$ has $\sigma(\mathcal{H}) = \mathcal{B}([0,1])$.
  $\mu = \mu_{U(0,1)}$, $f : u \mapsto 1-u$. Then $f_*\mu = \mu$, because

  $$f_*\mu([0,b)) = \mu(f^{-1}([0,b))) = \mu([1-b,1]) = 1-(1-b) = b.$$

- $\Omega = \mathbb{R}$, $y \in \mathbb{R}$, $f_y : x \mapsto x + y$
  λ Lebesgue measure. Then $(f_y)_*\lambda = \lambda$, because

  $$(f_y)_*\lambda([a,b]) = \lambda(f_y^{-1}([a,b])) = \lambda([a-y, b-y]) = b-a.$$

- $\Omega = [0,1], \Omega' = \mathbb{R}_+$, $f : x \mapsto -\frac{1}{\lambda}\log(x)$ for $\lambda > 0$
  $\mu = \mu_{U(0,1)}$. Then, $f_*\mu = \mu_{\exp(\lambda)}$, because for $x \ge 0$

  $$f_*\mu([0,x]) = \mu(f^{-1}([0,x])) = \mu([e^{-\lambda x}, 1]) = 1 - e^{-\lambda x}.$$

Measure Theory for Probabilists
9. Approximation of measurable functions

Peter Pfaffelhuber

January 14, 2024

Image measures

- If $\mathcal{F}'$ is a σ-field on Ω', and $f : \Omega \to \Omega'$. Then,

$$\sigma(f) := \{f^{-1}(A') : A' \in \mathcal{F}'\} \text{ is a } \sigma\text{-field on } \Omega.$$

- Definition 2.23: $(\Omega, \mathcal{F}, \mu)$ measure space, $(\Omega', \mathcal{F}')$ measurable space, $f : \Omega \to \Omega'$ with $\sigma(f) \subseteq \mathcal{F}$. Then,

$$\mathcal{F}' \ni A' \mapsto f_*\mu(A') := \mu(f^{-1}(A')) = \mu(f \in A')$$

  is the *image measure* of f under μ.
  If $\mathbb{P}$ is a probability measure, we call $X_*\mu$ the distribution of X under $\mathbb{P}$.
- Proposition 2.25: $f_*\mu$ is a measure on $\mathcal{F}'$.

Lemma 3.2

- $(\Omega', \mathcal{F}')$ measurable space, $f : \Omega \to \Omega'$, $\mathcal{C}' \subseteq \mathcal{F}'$ with $\sigma(\mathcal{C}') = \mathcal{F}'$. Then $\sigma(f^{-1}(\mathcal{C}')) = f^{-1}(\sigma(\mathcal{C}'))$.
- Proof: '$\subseteq$': $f^{-1}(\sigma(\mathcal{C}'))$ is a σ-algebra. So,

$$\sigma(f^{-1}(\mathcal{C}')) \subseteq \sigma(f^{-1}(\sigma(\mathcal{C}'))) = f^{-1}(\sigma(\mathcal{C}'))$$

'$\supseteq$': define

$$\widetilde{\mathcal{F}}' = \{A' \in \sigma(\mathcal{C}') : f^{-1}(A') \in \sigma(f^{-1}(\mathcal{C}'))\} \subseteq \sigma(\mathcal{C}').$$

Again, $\widetilde{\mathcal{F}}'$ is a σ-algebra and $\mathcal{C}' \subseteq \widetilde{\mathcal{F}}' \subseteq \sigma(\mathcal{C}')$. Thus, $\widetilde{\mathcal{F}}' = \sigma(\mathcal{C}')$. For $A' \in \sigma(\mathcal{C}')$, we find

$$f^{-1}(A') \in \sigma(f^{-1}(\mathcal{C}')),$$

which is equivalent to $f^{-1}(\sigma(\mathcal{C}')) \subseteq \sigma(f^{-1}(\mathcal{C}'))$.

Definition 3.3

- $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$ measurable spaces and $f : \Omega \to \Omega'$.
  1. f is *$\mathcal{F}/\mathcal{F}'$-measurable* if $f^{-1}(\mathcal{F}') \subseteq \mathcal{F}$. We define $\sigma(f) := f^{-1}(\mathcal{F}')$ the σ-algebra generated by f.
  2. If $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space and $X : \Omega \to \Omega'$ measurable, then X is called an *Ω'-valued random variable*. The image measure $X_*\mathrm{P}$ from Definition 2.23 is called the *distribution of X*.
  3. If $(\Omega', \mathcal{F}') = (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$, and f is $\mathcal{F}/\mathcal{F}'$-measurable, we say that f is (Borel-)measurable.
  4. If $f = 1_A$ for $A \subseteq \Omega$, then f is called *indicator function*. If $f = \sum_{k=1}^{n} c_k 1_{A_k}$ for $c_1, \dots, c_n \in \overline{\mathbb{R}}$ pairwise different and $A_1, \dots, A_n \subseteq \Omega$, then f is called *simple*.

Examples

- $f : \omega \mapsto \omega$ is measurable, since $f^{-1}(\mathcal{F}) = \mathcal{F}$.
- $(\Omega, \mathcal{O})$ and $(\Omega'.\mathcal{O}')$ topological spaces, $f : \Omega \to \Omega'$ continuous. Then f is measurable.
  Indeed: Since $f^{-1}(\mathcal{O}') \subseteq \mathcal{O}$. From Lemma 3.2,
  $$f^{-1}(\mathcal{B}(\Omega')) = f^{-1}(\sigma(\mathcal{O}')) = \sigma(f^{-1}(\mathcal{O}') \subseteq \sigma(\mathcal{O}) = \mathcal{B}(\Omega).$$
- A function $f : \Omega \to \{0,1\}$ is measurable if and only if $f^{-1}(\{1\}) \in \mathcal{F}$. Then, $\sigma(f) = \{\emptyset, f^{-1}(\{1\}), (f^{-1}(\{1\}))^c, \Omega\}$.
- For a non-measurable set/function, see Example 2.27 in the manuscript.

Examples for random variables

- (E, r) metric space, X an E-valued random variable on some probability space, Y an E-valued random variable on another probability space. If $X_*\mathrm{P} = Y_*\mathrm{Q}$, X and Y are *identically distributed* and we write $X \sim Y$.
- Let $(X_i)_{i\in I}$ family of random variables on a probability space. The distribution of $((X_i)_{i\in I})_*\mathrm{P}$ is called the *joint distribution of* $(X_i)_{i\in I}$.

Lemma 3.6

- If $\mathcal{C}' \subseteq \mathcal{F}'$ with $\mathcal{F}' = \sigma(\mathcal{C}')$, then $f : \Omega \to \Omega'$ is $\mathcal{F}/\mathcal{F}'$-measurable if and only if $f^{-1}(\mathcal{C}') \subseteq \mathcal{F}$.
- If $f : \Omega \to \Omega'$ is measurable and $g : \Omega' \to \Omega''$ is measurable, then $g \circ f : \Omega \to \Omega''$ is measuarble.
- A real-valued function f (i.e. $f : \Omega \to \mathbb{R}$) is measurable (with respect to $\mathcal{F}/\mathcal{B}(\mathbb{R})$) if and only if $\{\omega : f(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{Q}$.
- A simple function $f = \sum_{k=1}^{n} c_k 1_{A_k}$ with pairwise different $c_1, \dots, c_n \in \overline{\mathbb{R}}$ and $A_1, \dots, A_n \subseteq \Omega$ is measurable if and only if $A_1, \dots, A_n \in \mathcal{F}$.
- Proof of 1.:
  $f^{-1}(\mathcal{F}') = f^{-1}(\sigma(\mathcal{C}')) = \sigma(f^{-1}(\mathcal{C}')) \subseteq \sigma(\mathcal{F}) = \mathcal{F}$. This means that f is $\mathcal{F}/\mathcal{F}'$-measurable.

Algebraic structures of measurability

- Lemma 3.7: Let $f, g, f_1, f_2, \ldots$ be measurable. Then, the following are measurable: fg, $af + bg$ for $a, b \in \mathbb{R}$, f/g if $g(\omega) \neq 0$ for all $\omega \in \Omega$,

$$\sup_{n\in\mathbb{N}} f_n, \quad \inf_{n\in\mathbb{N}} f_n, \quad \limsup_{n\to\infty} f_n, \quad \liminf_{n\to\infty} f_n.$$

- In particular, $f^+, f^-, |f|$ are measurable.
- Proof: Consider $\psi(\omega) := (f(\omega), g(\omega))$ measruable. Then, $(x, y) \mapsto ax + by$, $(x, y) \mapsto xy$, $(x, y) \mapsto x/y$ are continuous, hence measurbale.
  2. for measurability of $\sup_{n\in\mathbb{N}} f_n$. Write, for $x \in \mathbb{R}$,

$$\left\{\omega : \sup_{n\in\mathbb{N}} f_n(\omega) \leq x\right\} = \bigcap_{n=1}^{\infty} \underbrace{\left\{\omega : f_n(\omega) \leq x\right\}}_{\in\mathcal{F}} \in \mathcal{F}.$$

Approximation by simple functions

- Theorem 3.9: $f : \Omega \to \overline{\mathbb{R}}_+$ measurable. Then there is $f_1, f_2, \dots : \Omega \to \overline{\mathbb{R}}$ of simple functions with $f_n \uparrow f$.
- Proof: Write

$$f_n(\omega) = n \wedge 2^{-n}[2^n f(\omega)] \uparrow f$$

by construction. Furthermore, $\omega \mapsto [2^n f(\omega)]$ is measurable according to Lemma 3.6.

Measure Theory for Probabilists
10. Defining the integral, and some properties

Peter Pfaffelhuber

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Outline

- Goal: For a measure μ, define for *many* functions $f : \Omega \to \mathbb{R}$

$$\mu[f] = \int f d\mu = \int f(\omega)\mu(d\omega).$$

- Initial step: For $f = 1_A$ for some $A \in \mathcal{F}$, define

$$\mu[f] := \mu(A).$$

- Definition 3.10: For $f = \sum_{k=1}^{m} c_k 1_{A_k}$ with $c_1, \ldots, c_m \geq 0, A_1, \ldots, A_m \in \mathcal{F}$, define

$$\mu[f] := \sum_{i=1}^{m} c_i \mu(A_i).$$

- Final step: f measurbale: use approximating sequence of simple functions.

Simple properties

- Lemma 3.12: f, g non-negative, simple functions and $\alpha \geq 0$. Then,

$$\mu[af + bg] = a\mu[f] + b\mu[g], \qquad f \leq g \Rightarrow \mu[f] \leq \mu[g].$$

- If $f = 1_A$ for $A \in \mathcal{F}$, note that f is in general not piecewise continuous. In particular, $\int f(x)dx$ does not exist in the sense of Riemann.

Integral of non-negative measurable functions

- Definition 3.14: $(\Omega, \mathcal{F}, \mu)$ measue space, $f : \Omega \to \overline{\mathbb{R}}_+$ measurable. Define

$$\mu[f] := \int f d\mu := \int f(\omega)\mu(d\omega)$$
$$:= \sup\{\mu[g] : g \text{ simple, non-negative, } g \leq f\}.$$

- Definition 3.17: $f : \Omega \to \overline{\mathbb{R}}$ measurable. Then f is said to be μ-integrable if $\mu[|f|] < \infty$,

$$\mu[f] := \int f(\omega)\mu(d\omega) := \int f d\mu := \mu[f^+] - \mu[f^-].$$

- For $A \in \mathcal{F}$ we also write

$$\mu[f, A] := \int_A f d\mu := \mu[f 1_A].$$

Proposition 3.16

- $f, g, f_1, f_2, \cdots : \Omega \to \overline{\mathbb{R}}_+$ measurable. Then,
  1. If $f \leq g$, then $\mu[f] \leq \mu[g]$.
  2. If
  $$f_n \uparrow f, \quad \text{then} \quad \mu[f_n] \uparrow \mu[f].$$
  3. If $a, b \geq 0$, then $\mu[af + bg] = a\mu[f] + b\mu[g]$.
- Proof: 1. clear.
  2. Since $f_1, f_2, \ldots \leq f$, $\lim_{n\to\infty} \mu[f_n] = \sup_{n\in\mathbb{N}} \mu[f_n] \leq \mu[f]$.
  For the reverse it suffices to show
  $$\mu[g] \leq \sup_{n\in\mathbb{N}} \mu[f_n]$$
  for all simple functions $g = \sum_{k=1}^m c_k 1_{A_k} \leq f$. Let $B_n^\varepsilon := \{f_n \geq (1-\varepsilon)g\}$. Since $f_n \uparrow f$ and $g \leq f$, $\bigcup_{n=1}^\infty B_n^\varepsilon = \Omega$
  $$\mu[f_n] \geq \mu[(1-\varepsilon)g 1_{B_n^\varepsilon}] = \sum_{k=1}^m (1-\varepsilon)c_k \mu(A_k \cap B_n^\varepsilon)$$
  $$\xrightarrow{n\to\infty} \sum_{k=1}^m (1-\varepsilon)c_k \mu(A_k) = (1-\varepsilon)\mu[g].$$

Some properties

- Define
$$\mathcal{L}^1(\mu) := \left\{ f : \Omega \to \overline{\mathbb{R}} : \mu[|f|^1] < \infty \right\}.$$
- Let $f, g \in \mathcal{L}^1(\mu)$. Then
  1. The integral is monotone, i.e.
  $$f \leq g \text{ almost everywhere} \qquad \Longrightarrow \qquad \mu[f] \leq \mu[g].$$
  In particular,
  $$|\mu[f]| \leq \mu[|f|].$$
  2. The integral is linear, so if $a, b \in \mathbb{R}$, then $af + bg \in \mathcal{L}^1(\mu)$ and
  $$\mu[af + bg] = a\mu[f] + b\mu[g].$$
  3. $g \in \mathcal{L}^1(f_*\mu)$, then $g \circ f \in \mathcal{L}^1(\mu)$ and
  $$\mu[g \circ f] = f_*\mu[g].$$
- Proof: 4. for simple, non-negative functions g. Note $g \circ f = \sum_{k=1}^m c_k 1_{f \in A'_k}$, hence
$$\mu[g \circ f] = \sum_{k=1}^m c_k \mu(f \in A'_k) = \sum_{k=1}^m c_k f_*\mu(A'_k) = f_*\mu[g].$$

Properties almost everywhere

- $f : \Omega \to \overline{\mathbb{R}}_+$ measurable.
  1. $f = 0$ almost everywhere iff $\mu[f] = 0$.
  2. If $\mu[f] < \infty$, then $f < \infty$ almost everywhere.
- Proof: 1. Let $N := \{f > 0\} \in \mathcal{F}$.
  '$\Rightarrow$': $\mu(N) = 0$, so

$$0 \leq \mu[f] = \mu[f, N] = \lim_{n\to\infty} \mu[n \wedge f, N] \leq \lim_{n\to\infty} \mu[n, N] = 0.$$

'$\Leftarrow$' Let $N_n := \{f \geq 1/n\}$, so $N_n \uparrow N$ and $nf \geq 1_{N_n}$, i.e.

$$0 = \mu[f] \geq \tfrac{1}{n}\mu(N_n).$$

This means that $\mu(N_n) = 0$ and therefore
$\mu(N) = \mu(\bigcup_{n=1}^\infty N_n) = 0$ by σ-sub-additivity of μ.
2. Let $A := \{f = \infty\}$. Since $f 1_{f\geq n} \geq n 1_{f \geq n}$,

$$\mu(A) = \mu[1_A] \leq \mu[1_{f\geq n}] \leq \tfrac{1}{n}\mu[f, 1_{f\geq n}] \leq \tfrac{1}{n}\mu[f] \xrightarrow{n\to\infty} 0.$$

Lebesgue and Riemann integral

- $f : \mathbb{R} \to \mathbb{R}$ be a piece-wise constant function, i.e.

$$f(x) = \sum_{j=-\infty}^{\infty} a_j 1_{[x_{j-1}, x_j)}(x)$$

  $f : [a,b] \to \mathbb{R}$ is *Riemann-integrable* if $\lambda[|f|] < \infty$ and there are piece-wise constant functions $f_n^- \leq f \leq f_n^+$ and $\lambda[f_n^+ - f_n^-] \xrightarrow{n\to\infty} 0$. Then, the Riemann integral and Lebesgue integral then coincide.
- $f : \mathbb{R} \to \mathbb{R}$ is called *Riemann-integrable* if $f 1_K$ is Riemann-integrable for all compact intervals $K \subseteq \mathbb{R}$ and $\lambda[f 1_{[-n,n]}]$ converges.

Riemann integrability

- Proposition 3.23: $f : [0,t] \to \mathbb{R}$ piecewise continuous. Then f is integrable, Riemann-integrable, and

$$\lambda[f] = \lim_{n\to\infty} \sum_{k=1}^{\infty} f(y_{n,k})(x_{n,k} - x_{n,k-1})$$

  for $0 = x_{n,0} \leq ... \leq x_{n,k_n} = t$ with $\max_k |x_{n,k} - x_{n,k-1}| \xrightarrow{n\to\infty} 0$ and any $x_{n,k-1} \leq y_{n,k} \leq x_{n,k}$.
- Proof for continuous f. Choose $\varepsilon_n \downarrow 0$ and $x_{n,0} \leq ... \leq x_{n,k_n}$ such that $K \subseteq [x_{n,0}, x_{n,k_n}]$ and $\max_{x_{n,k-1} \leq y < x_{n,k}} |f(x_{n,k-1}) - f(y)| < \varepsilon_n$. Then, find piecewise constant f_n^+, f_n^- with $f_n^- \leq f \leq f_n^+$ and $||f_n^+ - f_n^-|| \leq \varepsilon_n$. Integrability and Riemann-integrability follows. The formula follows from uniform approximation of the function f.

Lebesgue and Riemann integral

- $f = 1_{[0,1]\cap\mathbb{Q}}$ is not Riemann-integrable.
- $f(t) = \frac{(-1)^{\lceil t\rceil+1}}{\lceil t\rceil}$. Then

$$\lambda[f 1_{[0,2n]}] = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$
$$= \sum_{k=1}^{n} \frac{1}{2k-1} - \frac{1}{2k} = \sum_{k=1}^{n} \frac{1}{(2k-1)2k}$$

So, f is Riemann-integrable. However

$$\lambda[|f|] = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

So, $|f|$ is not integrable, hence f is not Lebesgue-integrable.

Measure Theory for Probabilists
11. Convergence results

Peter Pfaffelhuber

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Outline

- Theorem 3.25 for Riemann integral:
  $f, f_1, f_2, \ldots : [a, b] \to \mathbb{R}$ be piecewise continuous with $f_n \xrightarrow{n\to\infty} f$ uniformly. Then
  $$\int_a^b f_n(x)dx \xrightarrow{n\to\infty} \int_a^b f(x)dx.$$
- Theorem 3.26, monotone convergence:
  $f_1, f_2, \cdots \in \mathcal{L}^1(\mu)$ and $f : \Omega \to \overline{\mathbb{R}}$ measurable with $f_n \uparrow f$ almost everywhere. Then,
  $$\lim_{n\to\infty} \mu[f_n] = \mu[f].$$
- Theorem 3.28, dominated convergence:
  $f, g, f_1, f_2, \cdots : \Omega \to \overline{\mathbb{R}}$ measurable with $|f_n| \leq g$ almost everywhere, $\lim_{n\to\infty} f_n = f$ almost everywhere, and $g \in \mathcal{L}^1(\mu)$. Then,
  $$\lim_{n\to\infty} \mu[f_n] = \mu[f].$$

Monotone Convergence

- Theorem 3.26, monotone convergence:
  $f_1, f_2, \dots \in \mathcal{L}^1(\mu)$ and $f : \Omega \to \overline{\mathbb{R}}$ measurable with $f_n \uparrow f$ almost everywhere. Then,

$$\lim_{n\to\infty} \mu[f_n] = \mu[f].$$

- Proof: $N \in \mathcal{F}$ be such that $\mu(N) = 0$ and $f_n(\omega) \uparrow f(\omega)$ for $\omega \notin N$. Set $g_n := (f_n - f_1)1_{N^c} \geq 0$. This means that $g_n \uparrow (f - f_1)1_{N^c} =: g$ and with Proposition 3.16.2,

$$\mu[f_n] = \mu[f_1] + \mu[g_n] \xrightarrow{n\to\infty} \mu[f_1] + \mu[g] = \mu[f].$$

Lemma von Fatou

- Theorem 3.27: $f_1, f_2, \dots : \Omega \to \overline{\mathbb{R}}_+$ measurable. Then,

$$\liminf_{n\to\infty} \mu[f_n] \geq \mu[\liminf_{n\to\infty} f_n].$$

- Proof: For all $k \geq n$, $f_k \geq \inf_{\ell\geq n} f_\ell$ and thus, for all n,

$$\inf_{k\geq n} \mu[f_k] \geq \mu[\inf_{\ell\geq n} f_\ell].$$

So,

$$\liminf_{n\to\infty} \mu[f_n] = \sup_{n\in\mathbb{N}} \inf_{k\geq n} \mu[f_k] \geq \sup_{n\in\mathbb{N}} \mu[\inf_{k\geq n} f_k] = \mu[\liminf_{n\to\infty} f_n]$$

by monotone convergence.

Dominated convergence

- Theorem 3.28: $f, g, f_1, f_2, \dots : \Omega \to \overline{\mathbb{R}}$ measurable with $|f_n| \leq g$ almost everywhere, $\lim_{n\to\infty} f_n = f$ almost everywhere, and $g \in \mathcal{L}^1(\mu)$. Then,

$$\lim_{n\to\infty} \mu[f_n] = \mu[f].$$

- Proof: Wlog, $|f_n| \leq g$ and $\lim_{n\to\infty} f_n = f$ everywhere. Use Fatou's lemma and $g - f_n, g + f \geq 0$, i.e.

$$\mu[g+f] \leq \liminf_{n\to\infty} \mu[g+f_n] = \mu[g] + \liminf_{n\to\infty} \mu[f_n],$$
$$\mu[g-f] \leq \liminf_{n\to\infty} \mu[g-f_n] = \mu[g] - \limsup_{n\to\infty} \mu[f_n].$$

After subtracting $\mu[g]$,

$$\mu[f] \leq \liminf_{n\to\infty} \mu[f_n] \leq \limsup_{n\to\infty} \mu[f_n] \leq \mu[f].$$

Example

- λ: Lebesgue measure, $f_n = 1/n$. Then $f_n \downarrow 0$, but

$$\liminf_{n\to\infty} \mu[f_n] = \infty > 0 = \mu[0] = \mu[\liminf_{n\to\infty} f_n].$$

Example

$|f_n| \le g \in \mathcal{L}^1(\mu)$ is necessary (here for λ Lebesgue measure)

- $f_n = n \cdot 1_{[0,1/n]} \xrightarrow{n\to\infty} \infty \cdot 1_0$. There is no $g \in \mathcal{L}^1(\lambda)$ with $f_n \le g$ and

$$\lim_{n\to\infty} \mu[f_n] = 1 \neq 0 = \mu[\lim_{n\to\infty} f_n].$$

- $f_n = n \cdot 1_{[0,1/n^2]} \xrightarrow{n\to\infty} \infty \cdot 1_0$. There is $f_n \le g \in \mathcal{L}^1(\lambda)$ with

$$\sup_{n\in\mathbb{N}} f_n(x) = \sup\{n : x \le 1/n^2\} = \left\lfloor \frac{1}{\sqrt{x}} \right\rfloor \le \frac{1}{\sqrt{x}} =: g(x),$$

and

$$\lim_{n\to\infty} \mu[f_n] = \lim_{n\to\infty} \tfrac{1}{n} = 0 = \mu[0] = \mu[\lim_{n\to\infty} f_n].$$

Measure Theory for Probabilists
12. Basics of $\mathcal{L}^p$-spaces

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February 6, 2024

Definition of an $\mathcal{L}^p$-space

- For $0 < p \leq \infty$, set

$$\mathcal{L}^p := \mathcal{L}^p(\mu) := \{f : \Omega \to \overline{\mathbb{R}} \text{ measurable with } ||f||_p < \infty\}$$

for

$$||f||_p := (\mu[|f|^p])^{1/p}, \qquad 0 < p < \infty \tag{1}$$

and

$$||f||_\infty := \inf\{K : \mu(|f| > K) = 0\}.$$

Hölder's inequality

- Proposition 4.2.1: f, g be measurable, $0 < p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then,

$$||fg||_r \leq ||f||_p ||g||_q \qquad \text{(Hölder inequality)}$$

- Proof: $p = \infty$ or $||f||_p = 0$, $||f||_p = \infty$, $||g||_q = 0$ or $||g||_q = \infty$: ok, so assume any other case and define

$$\widetilde{f} := \frac{f}{||f||_p}, \qquad \widetilde{g} = \frac{g}{||g||_q}.$$

To show $||\widetilde{f}\widetilde{g}||_r \leq 1$. Convexity of the exponential function:

$$(xy)^r = \exp\left(\tfrac{r}{p} p \log x + \tfrac{r}{q} q \log y\right) \leq \tfrac{r}{p} x^p + \tfrac{r}{q} y^q,$$

and thus

$$||\widetilde{f}\widetilde{g}||_r^r = \mu[(\widetilde{f}\widetilde{g})^r] \leq \tfrac{r}{p}\mu[\widetilde{f}^p] + \tfrac{r}{q}\mu[\widetilde{g}^q] = 1.$$

Minkowski's inequality

- Proposition 4.2.2: For $1 \leq p \leq \infty$,

$$||f + g||_p \leq ||f||_p + ||g||_p.$$

- Proof: $p = 1$, $p = \infty$ clear. Else, let $q = p/(p-1)$ and $r = 1/p + 1/q = 1$, so Hölder's inequality gives

$$\begin{aligned} ||f + g||_p^p &\leq \mu[|f| \cdot |f + g|^{p-1}] + \mu[|g| \cdot |f + g|^{p-1}] \\ &\leq ||f||_p \cdot ||(f + g)^{p-1}||_q + ||g||_p \cdot ||(f + g)^{p-1}||_q \\ &= (||f||_p + ||g||_p) \cdot ||f + g||_p^{p-1}, \end{aligned}$$

since

$$\begin{aligned} ||(f + g)^{p-1}||_q = ||(f + g)^{q(p-1)}||_1^{1/q} = ||(f + g)^p||_1^{(p-1)/p} \\ = ||f + g||_p^{p-1}. \end{aligned}$$

Dividing by $||f + g||_p^{p-1}$ gives the result.

$p \mapsto \mathcal{L}^p$ is decreasing

- μ finite, $1 \leq r < q \leq \infty$. Then $\mathcal{L}^q(\mu) \subseteq \mathcal{L}^r(\mu)$.
- Counterexample for μ infinite: λ Lebesgue measure, $f : x \mapsto \frac{1}{x} \cdot 1_{x>1}$. Then $f \in \mathcal{L}^2(\lambda)$, but $f \notin \mathcal{L}^1(\lambda)$.
- Proof: $q = \infty$ clear; otherwise since $||1||_p < \infty$,

$$||f||_r = ||1 \cdot f||_r \leq ||1||_p \cdot ||f||_q < \infty$$

for $\frac{1}{p} = \frac{1}{r} - \frac{1}{q} > 0$

$\mathcal{L}^p$-convergence

- Definition 4.6: $f_1, f_2, \dots$ in $\mathcal{L}^p(\mu)$ *converges to* $f \in \mathcal{L}^p(\mu)$ iff
$$||f_n - f||_p \xrightarrow{n\to\infty} 0.$$
We write $f_n \xrightarrow{n\to\infty}_{\mathcal{L}^p} f$.
- Proposition 4.7: μ be finite, $1 \leq r < q \leq \infty$ and $f, f_1, f_2, \dots \in \mathcal{L}^q$. If $f_n \xrightarrow{n\to\infty}_{\mathcal{L}^q} f$, then also $f_n \xrightarrow{n\to\infty}_{\mathcal{L}^r} f$.
- Proof: clear since $||f - g||_r \leq ||f - g||_q$.

Completeness of $\mathcal{L}^p$

- Proposition 4.8: $p \geq 1, f_1, f_2, \ldots$ be a Cauchy sequence in $\mathcal{L}^p$. Then there is $f \in \mathcal{L}^p$ with $||f_n - f||_p \xrightarrow{n\to\infty} 0$.
- Proof: $\varepsilon_1, \varepsilon_2, \ldots$ summable. There is n_k for each k with $||f_m - f_n||_p \leq \varepsilon_k$ for all $m, n \geq n_k$. In particular,

$$\sum_{k=1}^{\infty} ||f_{n_{k+1}} - f_{n_k}||_p \leq \sum_{k=1}^{\infty} \varepsilon_k < \infty.$$

Monotone convergence and Minkowski give

$$\Big|\Big| \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}| \Big|\Big|_p \leq \sum_{k=1}^{\infty} ||f_{n_{k+1}} - f_{n_k}||_p < \infty.$$

In particular $\sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}| < \infty$ almost everywhere, i.e. for almost all $\omega \in \Omega$, the sequence $f_{n_1}(\omega), f_{n_2}(\omega), \ldots$ is Cauchy in $\mathbb{R}$, hence converges to some f. Fatou gives

$$||f_n - f||_p \leq \liminf_{k\to\infty} ||f_{n_k} - f_n||_p \leq \sup_{m\geq n} ||f_m - f_n||_p \xrightarrow{n\to\infty} 0,$$

i.e. $f_n \xrightarrow{n\to\infty}_{\mathcal{L}^p} f$.

Measure Theory for Probabilists
13. The space $\mathcal{L}^2$

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A scalar product

- Apparently, $\langle ., . \rangle : \mathcal{L}^2 \times \mathcal{L}^2 \to \mathbb{R}$, given by
$$\langle f, g \rangle := \mu[fg],$$
is bi-linear, symmetric and positive semi-definite.
- Complete normed spaces with a scalar product are called Hilbert spaces. So, $\mathcal{L}^2$ is a Hilbert space.
- Write $f \perp g$ iff $\mu[fg] = 0$

Parallelogram identity

- Lemma 4.9: For $f, g \in \mathcal{L}^2$,

$$||f+g||^2 + ||f-g||^2 = 2||f||^2 + 2||g||^2.$$

- Proof:

$$\begin{aligned} ||f+g||^2 + ||f-g||^2 &= \langle f+g, f+g\rangle + \langle f-g, f-g\rangle \\ &= 2\langle f, f\rangle + 2\langle g, g\rangle = 2||f||^2 + 2||g||^2. \end{aligned}$$

Decomposition

- Proposition 4.10: M closed, linear subspace of $\mathcal{L}^2$ and $f \in \mathcal{L}^2$. Then, there is an almost everywhere unique decomposition $f = g + h$ with $g \in M, h \perp M$.
- Proof: For $f \in \mathcal{L}^2$, define $d_f := \inf_{g \in M}\{||f - g|||\}$. Choose $g_1, g_2, \ldots$ with $||f - g_n|| \xrightarrow{n \to \infty} d_f$. Then

$$4d_f^2 + ||g_m - g_n||^2 \leq ||2f - g_m - g_n||^2 + ||g_m - g_n||^2$$
$$= 2||f - g_m||^2 + 2||f - g_n||^2 \xrightarrow{m,n \to \infty} 4d_f^2.$$

Thus $||g_m - g_n||^2 \xrightarrow{m,n\to\infty} 0$, i.e. $||g_n - g|| \xrightarrow{n \to \infty} 0$ for some $g \in M$ with $||f - g|| = d_f$. For $t > 0, l \in M$,

$$d_f^2 \leq ||f - g + tl||^2 = d_f^2 + 2t\langle f - g, l\rangle + t^2||l||^2.$$

Since this applies to all t, $\langle f - g, l\rangle = 0$, i.e. $f - g \perp M$. Uniqueness: Let $f = g + h = g' + h'$. Then, $g - g' \in M$ as well as $g - g' = h - h' \perp M$, i.e. $g - g' \perp g - g'$. This means $||g - g'|| = \langle g - g', g - g'\rangle = 0$, i.e. $g = g'$.

Theorem of Riesz-Fréchet

- Proposition 4.11: $F : \mathcal{L}^2 \to \mathbb{R}$ is continuous and linear iff there exists some $h \in \mathcal{L}^2$ with

$$F(f) = \langle f, h \rangle, \qquad f \in \mathcal{L}^2.$$

Then, $h \in \mathcal{L}^2$ is unique.

- Proof: '$\Leftarrow$' linearity clear. Continuity:

$$|\langle |f - f'|, h \rangle| \leq ||f - f'|| \cdot ||h||.$$

For uniqueness, let $\langle f, h_1 - h_2 \rangle = 0$ for all $f \in \mathcal{L}^2$; in particular, with $f = h_1 - h_2$

$$||h_1 - h_2||^2 = \langle h_1 - h_2, h_1 - h_2 \rangle = 0,$$

thus $h_1 = h_2$ μ-almost everywhere.

Theorem of Riesz-Fréchet

- Proposition 4.11: $F : \mathcal{L}^2 \to \mathbb{R}$ is continuous and linear iff there exists some $h \in \mathcal{L}^2$ with

$$F(f) = \langle f, h \rangle, \qquad f \in \mathcal{L}^2.$$

  Then, $h \in \mathcal{L}^2$ is unique.
- Proof: '$\Rightarrow$': For $F = 0$ choose $h = 0$. For $F \neq 0$, $M = F^{-1}\{0\}$ is closed and linear, so for $f' \in \mathcal{L}^2 \setminus M$, write $f' = g' + h'$ with $g' \in M$ and $h' \perp M$ and $F(h') = F(f') - F(g') = F(f') \neq 0$. Set $h'' = \frac{h'}{F(h')}$, so that $h'' \perp M$ and $F(h'') = 1$ and for $f \in \mathcal{L}^2$

$$F(f - F(f)h'') = F(f) - F(f)F(h'') = 0.$$

  i.e. $f - F(f)h'' \in M$, in particular $\langle F(f)h'', h'' \rangle = \langle f, h'' \rangle$ and

$$F(f) = \frac{1}{||h''||^2} \cdot \langle F(f)h'', h'' \rangle = \frac{1}{||h''||^2} \cdot \langle f, h'' \rangle = \langle f, \frac{h''}{||h''||^2} \rangle.$$

  Now, the assertion follows with $h := \frac{h''}{||h''||^2}$.

Measure Theory for Probabilists
14. Theorem of Radon-Nikodým

Peter Pfaffelhuber

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Theorem of Radon-Nikodým

- Corollary 4.17: μ, ν be σ-finite measures. Then, ν has a density with respect to μ if and only if $\nu \ll \mu$.

- Theorem 4.16 (Lebesgue decomposition theorem): μ, ν be σ-finite measures. Then ν can be written uniquely as

  $$\nu = \nu_a + \nu_s \qquad \text{with} \qquad \nu_a \ll \mu, \nu_s \perp \mu.$$

  The measure ν_a has a density with respect to μ that is μ-almost everywhere finite.

Absolute continuity

- Definition 4.13: ν has a *density* f with respect to μ if for all $A \in \mathcal{F}$,
$$\nu(A) = \mu[f; A].$$
We write $f = \frac{d\nu}{d\mu}$ and $\nu = f \cdot \mu$.
- ν is *absolutely continuous with respect to* μ if all μ-zero sets are also ν-zero sets. We write $\nu \ll \mu$. If both $\nu \ll \mu$ and $\mu \ll \nu$, then μ and ν are called *equivalent*.
- μ and ν are called *singular* if there is an $A \in \mathcal{F}$ with $\mu(A) = 0$ and $\nu(A^c) = 0$. We write $\mu \perp \nu$.

Chain rule

- Lemma 4.14: Let μ be a measure on $\mathcal{F}$.
  1. Let ν be a σ-finite measure. If g_1 and g_2 are densities of ν with respect to μ, then $g_1 = g_2$, μ-almost everywhere.
  2. Let $f : \Omega \to \mathbb{R}_+$ and $g : \Omega \to \mathbb{R}$ be measurable. Then,
  $$(f \cdot \mu)[g] = \mu[fg],$$
  if one of the two sides exists.
- Proof for finite μ: 1. Set $A := \{g_1 > g_2\}$. Since both g_1 and g_2 are densities of ν with respect to μ,
  $$0 = \nu(A) - \nu(A) = \mu[g_1 - g_2; A].$$
  Since only $g_1 > g_2$ is possible on A, $g_1 = g_2$ is $1_A\mu$-almost everywhere.
  2. For $g = 1_A$ with $A \in \mathcal{F}$, write
  $$(f \cdot \mu)[g] = (f \cdot \mu)(A) = \mu[f, A] = \mu[f 1_A] = \mu[fg].$$
  This extends up to the general case.

Examples

- For $\mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}_+$

$$f_{N(\mu,\sigma^2)}(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

  and λ is the one-dimensional Lebesgue measure. Then, $f_{N(\mu,\sigma^2)} \cdot \lambda$ is a *normal distribution*.

- For $\gamma \geq 0$, let

$$f_{\exp(\gamma)}(x) := 1_{x\geq 0} \cdot \gamma e^{-\gamma x}.$$

  Then, $f_{\exp(\gamma)} \cdot \lambda$ is called *exponential distribution with parameter* γ. From the chain rule,

$$\mathrm{E}[X] = f_{\exp(\gamma)} \cdot \lambda[\mathrm{id}] = \int_0^\infty \gamma e^{-\gamma x} x dx = \ldots = \frac{1}{\gamma}.$$

- Let μ be the counting measure on $\mathbb{N}_0$ and

$$f(k) = e^{-\gamma} \frac{\gamma^k}{k!}, \qquad k = 0, 1, 2, \ldots.$$

  Then $f \cdot \mu$ is the Poisson distribution for the parameter γ.

Theorem 4.16

- Let μ, ν be σ-finite measures. Then ν can be written uniquely as
$$\nu = \nu_a + \nu_s \qquad \text{with} \qquad \nu_a \ll \mu, \nu_s \perp \mu.$$
The measure ν_a has a density with respect to μ that is μ-almost everywhere finite.
- Proof for finite μ, ν. The map
$$\begin{cases} \mathcal{L}^2(\mu+\nu) & \to \mathbb{R}) \\ f & \mapsto \nu[f] \end{cases}$$
is continuous. By Riesz-Frechet, there is $h \in \mathcal{L}^2(\mu+\nu)$ with
$$\nu[f] = (\mu+\nu)[fh], \qquad \nu[f(1-h)] = \mu[fh], \qquad f \in \mathcal{L}^2(\mu+\nu).$$
For $f = 1_{\{h<0\}}$ and $f = 1_{\{h>1\}}$, we find
$$0 \le \nu\{h<0\} = (\mu+\nu)[h; h<0] \le 0,$$
$$0 \le \mu[h; \{h>1\}] = \nu[1-h; \{h>1\} \le 0.$$

Theorem 4.16

- Let μ, ν be σ-finite measures. Then ν can be written uniquely as
$$\nu = \nu_a + \nu_s \qquad \text{with} \qquad \nu_a \ll \mu, \nu_s \perp \mu.$$
The measure ν_a has a density with respect to μ that is μ-almost everywhere finite.
- Proof: Let $E := h^{-1}\{1\}$, and $f = 1_E$. Then,
$$\mu(E) = \mu[h; E] = \nu[1 - h; E] = 0.$$
Define $\nu = \nu_a + \nu_s$ and $\nu_s \perp \mu$ using
$$\nu_a(A) = \nu(A \setminus E), \qquad \nu_s(A) = \nu(A \cap E),$$
To show: $\nu_a \ll \mu$, so choose $A \in \mathcal{F}$ with $\mu(A) = 0$, so
$$\nu[1 - h; A \setminus E] = \mu[h; A \setminus E] = 0.$$
Since $h < 1$ on $A \setminus E$, $\nu_a(A) = \nu(A \setminus E) = 0$, i.e. $\nu_a \ll \mu$.

Theorem 4.16

- Let μ, ν be σ-finite measures. Then ν can be written uniquely as
$$\nu = \nu_a + \nu_s \qquad \text{with} \qquad \nu_a \ll \mu, \nu_s \perp \mu.$$
The measure ν_a has a density with respect to μ that is μ-almost everywhere finite.
- Proof: Define $\nu = \nu_a + \nu_s$ and $\nu_s \perp \mu$ using
$$\nu_a(A) = \nu(A \setminus E), \qquad \nu_s(A) = \nu(A \cap E),$$
To show: $g := \frac{h}{1-h} 1_{\Omega \setminus E}$ is the density of ν_a with respect to μ:
$$\mu[g; A] = \mu\Big[\frac{h}{1-h}; A \setminus E\Big] = \nu(A \setminus E) = \nu_a(A).$$
Uniqueness: let $\nu = \nu_a + \nu_s = \widetilde{\nu}_a + \widetilde{\nu}_s$ Choose $A, \widetilde{A} \in \mathcal{A}$ with $\nu_s(A) = \mu(A^c) = \widetilde{\nu}_s(\widetilde{A}) = \mu(\widetilde{A}^c) = 0$. Then,
$$\nu_a = 1_{A \cap \widetilde{A}} \cdot \nu_a = 1_{A \cap \widetilde{A}} \cdot \nu = 1_{A \cap \widetilde{A}} \cdot \widetilde{\nu}_a = \widetilde{\nu}_a.$$

Corollary 4.17

- Let μ, ν be σ-finite measures. Then, ν has a density with respect to μ if and only if $\nu \ll \mu$.
- Proof: '$\Rightarrow$': clear. '$\Leftarrow$': Lebesgue decomposition Theorem, there is a unique decomposition $\nu = \nu_a + \nu_s$ with $\nu_a \ll \mu, \nu_s \perp \mu$. Since $\nu \ll \mu$, $\nu_s = 0$ must apply and therefore $\nu = \nu_a$. In particular, the density of ν exists with respect to μ.

Measure Theory for Probabilists
15. Set systems on product spaces

Peter Pfaffelhuber

March 1, 2024

Product spaces

- For an index set I and a family of sets $(\Omega_i)_{i\in I}$, define the product space

$$\Omega := \bigtimes_{i\in I} \Omega_i := \{(\omega_i)_{i\in I} : \omega_i \in \Omega_i\}$$

For $H \subseteq J \subseteq I$, define projections

$$\pi_H^J : \bigtimes_{i\in J} \Omega_i \to \bigtimes_{i\in H} \Omega_i,$$

and $\pi_H := \pi_H^I$ and $\pi_i := \pi_{\{i\}}$, $i \in I$.

Topology on product spaces

- Definition 5.1: Let $(\Omega_i, \mathcal{O}_i)_{i \in I}$ be a family of topologcal spaces. Then,

$$\mathcal{O} := \mathcal{O}(\mathcal{C}), \qquad \mathcal{C} := \left\{ A_i \times \bigtimes_{j \in I, j \neq i} \Omega_j ; i \in I, A_i \in \mathcal{O}_i \right\}$$

  is called the *product topology* on Ω.
- All $\pi_i, i \in I$ are continuous with respect to the product topology.
  Indeed, for $A_i \in \mathcal{O}_i$,

$$\pi_i^{-1}(A_i) = A_i \times \bigtimes_{I \ni j \neq i} \Omega_j \in \mathcal{C} \subseteq \mathcal{O}.$$

The product σ-algebra

- Definition 5.3: Let $(\Omega_i, \mathcal{F}_i)_{i\in I}$ be a family of measurable spaces. Then,

$$\bigotimes_{i\in I} \mathcal{F}_i := \sigma(\mathcal{E}), \qquad \mathcal{E} := \left\{ A_i \times \bigtimes_{j\in I, j\neq i} \Omega_j : i \in I, A_i \in \mathcal{F}_i \right\}$$

  is the *product-σ-algebra* on Ω.
  We denote the Borel σ-algebra of $\mathcal{O}$ by $\mathcal{B}(\Omega)$.
- Projections are measurable.
- Lemma 5.5: Let $\mathcal{F}_i = \mathcal{B}(\Omega_i)$. For arbitrary I, we have $\bigotimes_{i\in I} \mathcal{B}(\Omega_i) \subseteq \mathcal{B}(\Omega)$. If I is countable and $(\Omega_i, \mathcal{O}_i)_{i\in I}$ are separable metric spaces, then $\mathcal{B}(\Omega) = \bigotimes_{i\in I} \mathcal{B}(\Omega_i)$.
- Proof: Clearly, $\mathcal{C} \subseteq \mathcal{O}(\mathcal{C})$, $\mathcal{C} \subseteq \mathcal{E}$ and $\mathcal{E} \subseteq \sigma(\mathcal{C})$. So,

$$\bigotimes_{i\in I} \mathcal{B}(\Omega_i) = \sigma(\mathcal{E}) = \sigma(\mathcal{C}) \subseteq \sigma(\mathcal{O}(\mathcal{C})) = \mathcal{B}(\Omega).$$

  If I is countable and all spaces are separable, every $A \in \mathcal{O}(\mathcal{C})$ is a countable union of sets in $\mathcal{C}$, so $\mathcal{O}(\mathcal{C}) \subseteq \sigma(\mathcal{C})$. Hence,

$$\sigma(\mathcal{O}(\mathcal{C})) \subseteq \sigma(\sigma(\mathcal{C})) = \sigma(\mathcal{C}).$$

Products of generators

- Lemma 5.7: Let $(\Omega_i, \mathcal{F}_i)$ be measurable spaces and $\Omega = \bigtimes_{i \in I} \Omega_i$.
    1. I finite, $\mathcal{H}_i$ semi-ring with $\sigma(\mathcal{H}_i) = \mathcal{F}_i$. Then

       $$\mathcal{H} := \left\{ \bigtimes_{i \in I} A_i : A_i \in \mathcal{H}_i, i \in I \right\}$$

       is semi-ring with $\sigma(\mathcal{H}) = \bigotimes_{i \in I} \mathcal{F}_i$.
    2. I arbitrary, $\mathcal{H}_i$ a $\cap$-stable generator of $\mathcal{F}_i$, $i \in I$. Then

       $$\mathcal{H} := \{ \bigtimes_{i \in J} A_i \times \bigtimes_{i \in I \setminus J} \Omega_i : J \subseteq_f I, A_i \in \mathcal{H}_i, i \in J \}$$

       is $\cap$-stable generator of $\bigotimes_{i \in I} \mathcal{F}_i$.

σ-algebra on $\mathbb{R}^d$

- Corollary 5.8: Let $\Omega = \mathbb{R}^d$. For $\underline{a}, \underline{b} \in \mathbb{R}^d$, denote
$$(\underline{a}, \underline{b}] = (a_1, b_1] \times \cdots \times (a_d, b_d].$$
Then,
$$\mathcal{H} := \{(\underline{a}, \underline{b}] : \underline{a}, \underline{b} \in \mathbb{Q}, \underline{a} \leq \underline{b}\}$$
is a semi-ring with $\sigma(\mathcal{H}) = \mathcal{B}(\mathbb{R}^d)$.
- Proof: $\mathcal{H}$ is a semi-ring that generates $\bigotimes_{i=1}^{d} \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^d)$

Measure Theory for Probabilists
16. Measures on product spaces

Peter Pfaffelhuber

March 1, 2024

Definition 5.9

$(\Omega_i, \mathcal{F}_i), i = 1, 2$ measurable spaces.

- $\kappa : \Omega_1 \times \mathcal{F}_2 \to \mathbb{R}_+$ is a *transition kernel from* $(\Omega_1, \mathcal{F}_1)$ *to* $(\Omega_2, \mathcal{F}_2)$ if
  (i) for all $\omega_1 \in \Omega_1$, the map $\kappa(\omega_1, .)$ is a measure on $\mathcal{F}_2$ and
  (ii) for all $A_2 \in \mathcal{F}_2$ $\kappa(., A_2)$ is $\mathcal{F}_1$-measurable.
- A transition kernel is called σ-finite if there is a sequence $\Omega_{21}, \Omega_{22}, \dots \in \mathcal{F}_2$ with $\Omega_{2n} \uparrow \Omega_2$ and $\sup_{\omega_1} \kappa(\omega_1, \Omega_{2n}) < \infty$ for all $n = 1, 2, \dots$
- It is called *stochastic kernel* or *Markov kernel* if for all $\omega_1 \in \Omega_1$ the map $\kappa(\omega_1, .)$ is a probability measure.

Example: Markov chain

- $\Omega = \{\omega_1, \dots, \omega_n\}$ finite and $P = (p_{ij})_{1 \le i,j \le n}$ with $p_{ij} \in [0,1]$ and $\sum_{j=1}^n p_{ij} = 1$. Then,

$$\kappa(\omega_i, .) := \sum_{j=1}^{n} p_{ij} \cdot \delta_{\omega_j}$$

  is a Markov kernel from $(\Omega, 2^\Omega)$ to $(\Omega, 2^\Omega)$.
- P is the transition matrix of a homogeneous, Ω-valued Markov chain.

- $(\Omega_i, \mathcal{F}_i), i = 1, 2$ be measurable spaces, μ a σ-finite measure on $\mathcal{F}_0$, κ a σ-finite transition kernel from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$
- Lemma 5.11: Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}_+$ be $\mathcal{F}_1 \otimes \mathcal{F}_2$ measurable. Then,

$$\omega_1 \mapsto \kappa(\omega_1, .)[f] := \int \kappa(\omega_1, d\omega_2) f(\omega_1, \omega_2)$$

  is $\mathcal{F}_1$-measurable.
- Theorem 5.12: There is exactly one σ-finite measure $\mu \otimes \kappa$ on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ with

$$\Big(\mu \otimes \kappa\Big)(A \times B) = \int_A \mu(d\omega_1)\Big(\int_B \kappa(\omega_1, d\omega_2)\Big).$$

Fubini's Theorem

- Theorem 5.13: $(\Omega_i, \mathcal{F}_i)$, μ, κ and $\mu \otimes \kappa$ as above. Let $f : \Omega_1 \to \Omega_2 \to \mathbb{R}_+$ measurable with respect to $\mathcal{F}_1 \otimes \mathcal{F}_2$. Then,

$$\int f d\big(\mu \otimes \kappa\big) = \int \mu(d\omega_1)\Big(\int \kappa(\omega_1, d\omega_2) f(\omega_1, \omega_2)\Big).$$

  Equality also applies if $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable with $\int |f| d\big(\mu \otimes \kappa\big) < \infty$.

- Corollary 5.14: Let $\Omega = \Omega_1 \times \Omega_2$ and $\mathcal{H}_i \subseteq 2^{\Omega_i}$ be a semi-ring, and $\mu_i : \mathcal{H}_i \to \mathbb{R}_+$ σ-finite and, σ-additive, $i = 1, 2$. Then there is exactly one measure $\mu_1 \otimes \mu_2$ on $\sigma(\mathcal{H}_1) \otimes \sigma(\mathcal{H}_2)$ with

$$\mu_1 \otimes \mu_2(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2).$$

  For $f : \Omega \to \mathbb{R}_+$ measurable, the value of the integral does not depend on the order of integration.

Definition and Example

- $\lambda^{\otimes d}$ is d-dimensional Lebesgue measure. Let

$$f(x,y) = \frac{xy}{(x^2+y^2)^2}.$$

Then, for every $x \in \mathbb{R}$

$$\int \lambda(dy) f(x,y) = 0,$$

since $f(x,.) \in \mathcal{L}^1(\lambda)$ and $f(x,y) = -f(x,-y)$. Therefore, iterated integrals are 0. However, $|f|$ is not integrable because f has a non-integrable pole in $(0,0)$.

Convolutions of measures 1

- Definition 5.17: Let μ_1, μ_2 be σ-finite measures on $\mathcal{B}(\mathbb{R})$ and $\mu_1 \otimes \mu_2$ their product measure. Let $S(x_1, x_2) := x_1 + x_2$. Then $S_*(\mu_1 \otimes \cdots \otimes \mu_n)$ is the *convolution* of μ_1, μ_2 and is denoted by $\mu_1 * \mu_2$.
- $\gamma_1, \gamma_2 \geq 0$, $\mu_{\text{Poi}(\gamma_1)}$ and $\mu_{\text{Poi}(\gamma_2)}$. Then,

$$\begin{aligned}\mu_{\text{Poi}(\gamma_1)} * \mu_{\text{Poi}(\gamma_2)} &= \sum_{m,n} 1_{m+n=k} e^{-(\gamma_1+\gamma_2)} \frac{\gamma_1^m \gamma_2^n}{m!n!} \cdot \delta_k \\ &= \sum_{m=0}^{k} e^{-(\gamma_1+\gamma_2)} \frac{\gamma_1^m \gamma_2^{k-m}}{m!(k-m)} \cdot \delta_k \\ &= e^{-(\gamma_1+\gamma_2)} \frac{(\gamma_1+\gamma_2)^k}{k!} \cdot \delta_k \sum_{m=0}^{k} \binom{k}{m} \frac{\gamma_1^m \gamma_2^{k-m}}{(\gamma_1+\gamma_2)^k} \\ &= \mu_{\text{Poi}(\gamma_1+\gamma_2)}.\end{aligned}$$

Convolutions of measures 2

- Lemma 5.19: λ measure on $\mathcal{B}(\mathbb{R})$, $\mu = f_\mu \cdot \lambda$ and $\nu = f_\nu \cdot \lambda$. Then,$\mu * \nu = f_{\mu*\nu} \cdot \lambda$ with

$$f_{\mu*\nu}(t) = \int f_\mu(s) f_\nu(t-s)\lambda(ds).$$

- $f_{N(\mu_1,\sigma_1^2)}$ and $f_{N(\mu_2,\sigma_2^2)}$. Let $\mu := \mu_1 + \mu_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$. Then, the density of $N(\mu_1, \sigma_1^2) * N(\mu_2, \sigma_2^2)$ is

$$\begin{aligned}
x \mapsto & \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2}} \int \exp\Big(- \frac{(y-\mu_1)^2}{2\sigma_1^2} - \frac{(x-y-\mu_2)^2}{2\sigma_2^2}\Big) dy \\
& = \cdots = \\
& = \frac{1}{2\pi\sigma} \int \exp\Big(- \frac{(\sigma y - \frac{\sigma_1}{\sigma_2}(x-\mu))^2}{2\sigma^2} - \frac{(x-\mu)^2\Big(\frac{\sigma^2}{\sigma_2^2} - \frac{\sigma_1^2}{\sigma_2^2}\Big)}{2\sigma^2}\Big) dy \\
& = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(- \frac{(x-\mu)^2}{2\sigma^2}\Big).
\end{aligned}$$

Measure Theory for Probabilists
17. Projective limits

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Purpose

- Let $X_1, X_2, \dots$ be coin tosses, i.e. random variables with values in $\{0, 1\}$. What is the joint distribution of $(X_1, X_2, \dots)$?
- Let $(X_t)_{t \in [0,\infty)}$ some random process. What is its distribution?

- $\to$ We need to consider probability measures on (uncountably) infinite product spaces!!
- We will do this using our usual construction with outer measures based on a projective family.
- Recall für $H \subseteq J$ the projection $\pi_H^J : \Omega^J \to \Omega^H$.

Projective family and limit

- $(\Omega, \mathcal{F})$ measurable space, I arbitrary.
- Definition 5.21: A family $(\mathrm{P}_J)_{J \subseteq_f I}$, where P_J is a probability measure on $\mathcal{F}^J := \mathcal{F}^{\otimes J}$, is called projective if

$$\mathrm{P}_H = (\pi_H^J)_* \mathrm{P}_J, \qquad H \subseteq J \subseteq_f I.$$

If there exists a measure P_I on $\mathcal{F}^I := \mathcal{F}^{\otimes I}$ with

$$\mathrm{P}_J = (\pi_J)_* \mathrm{P}_I, \qquad J \subseteq_f I,$$

then we call P_I its projective limit and write

$$\mathrm{P}_I = \varprojlim_{J \subseteq_f I} \mathrm{P}_J.$$

Uniqueness

- Remark 5.23: Projective limits are unique:
  Indeed:

  $$\mathcal{H}' := \Big\{ \bigtimes_{i \in J} A_i \times \bigtimes_{i \in I \setminus J} \Omega_i, A_i \in \mathcal{F}_i, i \in J \subseteq_f I \Big\},$$

  is a $\cap$-stable generator of $\mathcal{F}^{\otimes I}$. If $\mathrm{P}_I = \varprojlim_{J \subseteq_f I} \mathrm{P}_J$. and $A = \bigtimes_{i \in J} A_i \times \bigtimes_{i \in I \setminus J} \Omega \in \mathcal{H}'$,

  $$\mathrm{P}_I(A) = \mathrm{P}_J\Big(\bigtimes_{i \in J} A_i \Big).$$

Existence

- Theorem 5.24: Let Ω be Polish and $(\mathrm{P}_J)_{J\subseteq_f I}$ a projective family. Then, the projective limit $\varprojlim_{J\subseteq_f I} \mathrm{P}_J$ exists.
- Proof: $\mathcal{H}'$ semi-ring as above. For $A = \times_{i\in J} A_i \times \times_{i\in I\setminus J} \Omega \in \mathcal{H}'$, define

$$\mu(A) := \mathrm{P}_J(\bigtimes_{i\in J} A_i)$$

and use the compact system

$$\mathcal{K} := \{\bigtimes_{j\in J} K_j \times \bigtimes_{i\in I\setminus J} \Omega : J \subseteq_f I, K_j \text{ compact}\} \subseteq \mathcal{H}.$$

To show: μ is inner regular with respect to $\mathcal{K}$.
Then. According to Theorem 2.10, μ is σ-additive.
Furthermore, $\mu(\Omega^I) = 1$, so μ can be uniquely extended to a measure P on $\sigma(\mathcal{H}) = \mathcal{F}^I$ according to Theorem 2.16.

Existence

- Theorem 5.24: Let Ω be Polish and $(\mathrm{P}_J)_{J\subseteq_f I}$ a projective family. Then, the projective limit $\varprojlim_{J\subseteq_f I} \mathrm{P}_J$ exists.
- To show: μ is inner regular with respect to $\mathcal{K}$.
  For $\varepsilon > 0$ and $j \in J$, there is $K_j \subseteq A_j$ cp with $\mathrm{P}_j(A_j \setminus K_j) < \varepsilon$. Then,

$$\begin{aligned}
&\mu\Big(\Big(\bigtimes_{i\in J} A_i \times \bigtimes_{i\in I\setminus J} \Omega\Big) \setminus \Big(\bigtimes_{i\in J} K_i \times \bigtimes_{i\in I\setminus J} \Omega\Big)\Big)\\
&= \mu\Big(\Big((\bigtimes_{i\in J} A_i) \setminus (\bigtimes_{i\in J} K_i)\Big) \times \bigtimes_{i\in I\setminus J} \Omega\Big)\\
&= \mathrm{P}_J\big(\big(\bigtimes_{j\in J} A_j\big) \setminus \big(\bigtimes_{j\in J} K_j\big)\big)\\
&\leq \mathrm{P}_J\Big(\bigcup_{j\in J} (A_j \setminus K_j) \times \bigtimes_{i\neq j} \Omega\Big)\\
&\leq \sum_{j\in J} \mathrm{P}_J\Big((A_j \setminus K_j) \times \bigtimes_{i\neq j} \Omega\Big) = \sum_{j\in J} \mathrm{P}_j(A_j \setminus K_j) \leq |J|\varepsilon.
\end{aligned}$$