universität freiburg

Measure theory for probabilists **Winter semester 2024**

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https://pfaffelh.github.io/hp/2024WS_measure_theory.html
https://www.stochastik.uni-freiburg.de/

Tutorial 6 - Set functions

Exercise 1 (4 Points).
Let λ be the Lebesgue-measure.

(a) Show that $\lambda(A) = 0$ where A is any finite set.

(b) Show that $\lambda(\mathbb{Q}) = 0$ (In general, the Lebesgue-measure of any countable set is 0).

(c) Let A be the Cantor set from Example 1.10. Compute $\lambda(A)$.

Exercise 2 (4 Points).

(a) let (Ω,r) is a metric space. Show that if a set $E \subseteq \Omega$ has positive outer measure, then there is a bounded subset of E that also has positive outer measure.

(b) Show that if E_1 and E_2 are measurable, then

$$\mu^*(E_1 \cup E_2) + \mu^*(E_1 \cap E_2) = \mu^*(E_1) + \mu^*(E_2).$$

Exercise 3 (4 Points).

(a) Let $\mu = \mu_{B(n,p)}$ be the binomial distribution with n trials and success probability p. Let $f : [0,n] \to [0,n]$ be defined by $f(k) = n - k$. Prove that $f_*\mu = \mu_{B(n,1-p)}$. Can you formulate a similar statement for the hypergeometric distribution?

(b) Let $f : \mathbb{R}_+ \to \mathbb{N}$ be given by $f(x) = \lceil x \rceil := \min\{n \in \mathbb{N} : n \geq x\}$, and $\mu = \mu_{\exp(\lambda)}$. Show that $f_*\mu$ is a geometric distribution and compute its success probability.

Exercise 4 (4 Points).

1. Prove that if $\mu^*(A) = 0$, then $\mu^*(A \cup B) = \mu^*(B)$.

2. Let (Ω,r) be a metric space, and μ^* the outer measure from Proposition 2.15, where $\mathcal{F}$ is the topology generated from (Ω,r). In addition, let A and B be bounded sets for which there is an $\alpha > 0$ such that $r(a,b) \geq \alpha$ for all $a \in A, b \in B$. Prove that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.