universitätfreiburg

Measure theory for probabilists **Winter semester 2024**

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https://pfaffelh.github.io/hp/2024WS_measure_theory.html
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Tutorial 5 - Set systems II

Exercise 1 (4 Points)**.**
Let $\mathcal{C} \subseteq 2^\Omega$. Show that $\mathcal{C} \subseteq \sigma(\mathcal{C})$

Solution.
Let $\mathcal{F}$ be any σ-field that contains $\mathcal{C}$. Since $\mathcal{F}$ is a σ-field and contains $\mathcal{C}$, it must include all the sets in $\mathcal{C}$. For every $A \in \mathcal{C}$, since A is in $\mathcal{F}$ (because $\mathcal{F}$ contains $\mathcal{C}$), we have $A \in \mathcal{F}$. Since this holds for any σ-field $\mathcal{F}$ that contains $\mathcal{C}$, we can conclude that:

$$A \in \bigcap\{\mathcal{F} \supseteq \mathcal{C} : \mathcal{F} \text{ is a } \sigma\text{-field}\}$$

which is exactly the definition of $\sigma(\mathcal{C})$. Thus, $A \in \sigma(\mathcal{C})$. Since A is arbitrary, $\mathcal{C} \subseteq \sigma(\mathcal{C})$

Exercise 2 (4 Points)**.**
Show that $\lambda(\mathcal{C})$ is a Dynkin-system.

Solution.
Let $\mathcal{C} \subseteq 2^\Omega$, and recall that:

$$\lambda(\mathcal{C}) := \bigcap\{\mathcal{D} \supseteq \mathcal{C} : \text{Dynkin system}\}$$

We need to show that $\lambda(\mathcal{C})$ satisfies the conditions in Definition 1.11:

(i) Containment of Ω: Since every Dynkin system $\mathcal{D}$ that contains $\mathcal{C}$ must contain Ω (by property (i) of Dynkin systems), it follows that:

$$\Omega \in \mathcal{D} \implies \Omega \in \lambda(\mathcal{C})$$

(ii) Set-Difference Stability: Let $A,B \in \lambda(\mathcal{C})$ with $A \subseteq B$. Since A,B belong to $\lambda(\mathcal{C})$, they are in all Dynkin systems $\mathcal{D}$ that contain $\mathcal{C}$. For any such Dynkin system $\mathcal{D}$, by the property of Dynkin systems, we have:

$$B \setminus A \in \mathcal{D}$$

Since this holds for all Dynkin systems containing $\mathcal{C}$, it follows that:

$$B \setminus A \in \lambda(\mathcal{C})$$

(iii) Closure under Countable Increasing Unions: Let $A_1, A_2, \ldots \in \lambda(\mathcal{C})$ such that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$. Each A_n is in every Dynkin system $\mathcal{D}$ that contains $\mathcal{C}$. By the property of Dynkin systems, since (A_n) is an increasing sequence, we have: $\bigcup_{n=1}^\infty A_n \in \mathcal{D}$. This holds for all Dynkin systems $\mathcal{D}$ containing $\mathcal{C}$, thus: $\bigcup_{n=1}^\infty A_n \in \lambda(\mathcal{C})$

Exercise 3 (4 Points)**.**

Let $\Omega = \{1,...,n\}$ for some even $n \in \mathbb{N}$ and $\mathcal{D}$ be the set of subsets of even cardinality. Show that $\mathcal{D}$ is a Dynkin system, but it is not a σ-algebra.

Solution.

Since $|\Omega|$ is even, $\Omega \in \mathcal{D}$ is even. If $A,B \in \mathcal{D}$ with $A \subset B$, then surely $|B \setminus A| = |B| - |A|$ is even and therefore $B \setminus A \in \mathcal{D}$. Finally, let $A_1,A_2,\ldots \in \mathcal{D}$ be an ascending sequence. Then, since $|\Omega| < \infty$, it also holds that $|\bigcup_{k\geq 1} A_k| \leq |\Omega| < \infty$. In particular, there exists n such that $\bigcup_{k\geq 1} A_k = \bigcup_{k=1}^{n} A_k = A_n \in \mathcal{D}$. $\mathcal{D}$ is therefore a Dynkin system according to Definition 1.11.

However, $\mathcal{D}$ cannot be $\cap$-stable and therefore cannot be a σ-algebra (see Table 1). This is because, since $|\mathcal{D}| \geq 2$ we can find three different ω_1, ω_2 ,ω_3 $\in \Omega$. Then $\{\omega_1, \omega_2\} \in \mathcal{D}$ and $\{\omega_2, \omega_3\} \in \mathcal{D}$ but not $\{\omega_1, \omega_2\} \cap \{\omega_2, \omega_3\} = \{\omega_2\}$.

Exercise 4 (3+1= 4 Points)**.**

(a) Prove that the intersection of rings is a ring and the intersection of σ-fields is a σ-field. Does the same hold for semi-rings/topologies?

(b) Give a counterexample that shows that, in general, the union of two σ-fields is not necessarily a σ-field.

Solution.

(ai) Let $\mathcal{R}_1,\mathcal{R}_2,\ldots$ be a collection of rings. We need to show that the intersection $\mathcal{R} = \bigcap_i \mathcal{R}_i$ is also a ring.

- **Closure under Union**: Let $A,B \in \mathcal{R}$. Then $A \in \mathcal{R}_i$ and $B \in \mathcal{R}_i$ for all i. Since each $\mathcal{R}_i$ is a ring, we have: $A \cup B \in \mathcal{R}_i$ for all i. Therefore, $A \cup B \in \mathcal{R}$.
- **Closure under Set Differences**: Let $A,B \in \mathcal{R}$. Then $A,B \in \mathcal{R}_i$ for all i. Since each $\mathcal{R}_i$ is a ring, we have: $A \setminus B \in \mathcal{R}_i$ for all i. Therefore, $A \setminus B \in \mathcal{R}$.

Thus, $\mathcal{R}$ is a ring.

(aii) Let $\mathcal{F}_1,\mathcal{F}_2,\ldots$ be a collection of σ-fields. We need to show that the intersection $\mathcal{F} = \bigcap_i \mathcal{F}_i$ is also a σ-field.

- **Containment of** Ω: Since $\Omega \in \mathcal{F}_i$ for all i, we have $\Omega \in \mathcal{F}$.
- **Closure under Complements**: Let $A \in \mathcal{F}$. Then $A \in \mathcal{F}_i$ for all i. Since each $\mathcal{F}_i$ is a σ-field, we have $A^c \in \mathcal{F}_i$ for all i. Therefore, $A^c \in \mathcal{F}$.
- **Closure under Countable Unions**: Let $A_1,A_2,\ldots \in \mathcal{F}$. Then $A_n \in \mathcal{F}_i$ for all i and for all n. Since each $\mathcal{F}_i$ is a σ-field, we have: $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_i$ for all i. Therefore, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Thus, $\mathcal{F}$ is a σ-field. However, the intersection of semi-rings is not necessarily a semi-ring. While semi-rings are closed under finite intersections and differences, the intersection of two semi-rings may not satisfy the closure properties required for semi-rings. For example, consider $X := \{1,2,3,4\}$ and $\mathcal{H}_1 := \{\emptyset,\{1\},\{4\},\{2,3\},\{1,2,3,4\}\}$ and $\mathcal{H}_2 := \{\emptyset,\{1\},\{2\},\{3,4\},\{1,2,3,4\}\}$. $\mathcal{H}_1$ and $\mathcal{H}_2$ are obviously semi-rings. However,

$$\mathcal{H}_1 \cap \mathcal{H}_2 = \{\emptyset,\{1\},\{1,2,3,4\}\}$$

is not a semi-ring. Lastly, the intersection of topologies is not necessarily a topology. A topology is closed under arbitrary unions and finite intersections, but the intersection of two topologies may not include all unions of sets from both topologies. Formally, by definition A.1(3), let $\mathcal{O}_1$ and $\mathcal{O}_2$ be topologies on a set Ω. Define the intersection of these topologies as:

$$\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2 = \{A \in 2^{\Omega} \mid A \in \mathcal{O}_1 \text{ and } A \in \mathcal{O}_2\}$$

Since $\emptyset, \Omega \in \mathcal{O}_1$ and $\mathcal{O}_2$, it follows that:

$$\emptyset \in \mathcal{O} \quad \text{and} \quad \Omega \in \mathcal{O}$$

Let $A, B \in \mathcal{O}$. Then $A \in \mathcal{O}_1$ and $A \in \mathcal{T}_2$, and similarly for B. Since both $\mathcal{O}_1$ and $\mathcal{T}_2$ are closed under finite intersections, we have:

$$A \cap B \in \mathcal{O}_1 \quad \text{and} \quad A \cap B \in \mathcal{O}_2$$

Let I be an arbitrary index set and $A_i \in \mathcal{O}$ for each $i \in I$. This means $A_i \in \mathcal{O}_1$ and $A_i \in \mathcal{O}_2$ for all i. Since both $\mathcal{O}_1$ and $\mathcal{O}_2$ are closed under arbitrary unions, we have:

$$\bigcup_{i \in I} A_i \in \mathcal{O}_1 \quad \text{and} \quad \bigcup_{i \in I} A_i \in \mathcal{O}_2$$

Therefore, $\bigcup_{i \in I} A_i \in \mathcal{O}$. Since the intersection $\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2$ satisfies all three properties of a topology, we conclude that the intersection of any collection of topologies on a set is also a topology.

(b) Let $X := \{a,b,c\}$ and $\mathcal{F}_1 := \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}\}$ and $\mathcal{F}_2 = \{\emptyset, \{b\}, \{a,c\}, \{a,b,c\}\}$. Then,

$$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\emptyset, \{a\}, \{b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

is not a σ-algebra since $\{a\} \cup \{b\} = \{a,b\} \notin \mathcal{F}_1 \cup \mathcal{F}_2$.