universität freiburg

Measure theory for probabilists — Winter semester 2024

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https://pfaffelh.github.io/hp/2024WS_measure_theory.html
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Tutorial 4 - Set systems I

Exercise 1 (4 Points)**.**
Let $\Omega = \{1, \dots, 5\}$. Determine the generated σ-algebra of:

(a) $\mathcal{E} := \{\{1,2,3,4\}\}$,

(b) $\mathcal{F} := \{\{1,2,3\},\{4\}\}$,

(c) $\mathcal{G} := \{1,2,3,4\}$,

(d) $\mathcal{H} := \emptyset$.

Solution.

(a) $\sigma(\mathcal{E}) = \{\emptyset,\Omega,\{1,2,3,4\},\{5\}\}$.

(b) $\sigma(\mathcal{F}) = \{\emptyset,\Omega,\{1,2,3\},\{4\},\{4,5\},\{1,2,3,5\},\{1,2,3,4\},\{5\}\}$

(c) $\mathcal{G}$ is not a set system. This means that there is no σ-algebra $\mathcal{A}$ on Ω with $\mathcal{G} \subset \mathcal{A}$ so there is no smallest one either. $\sigma(\mathcal{G})$ is therefore not defined.

(d) However, an empty set system is not a problem for the definition. The smallest σ-algebra without any constraints is always the trivial $\sigma(\emptyset) = \{\emptyset,\Omega\}$.

Exercise 2 (2+2=4 Points)**.**
Let $f : X \to Y$ be a mapping and let $\mathcal{A} \subset \mathcal{P}(X)$, $\mathcal{B} \subset \mathcal{P}(Y)$. Decide (with reasons) whether the following statements are correct:

(a) If $\mathcal{A}$ is a σ-algebra, then $\{B \subset Y \mid f^{-1}(B) \in \mathcal{A}\}$ is also a σ-algebra.

(b) If $\mathcal{B}$ is a σ-algebra, then $f^{-1}(\mathcal{B}) := \{f^{-1}(B) \mid B \in \mathcal{B}\}$ is also a σ-algebra.

Solution.

(a) We claim that the statement is true and check the properties of the σ-algebra:

  (i) $f^{-1}(Y) = X \in \mathcal{A}$, i.e. $Y \in \mathcal{B}$.

  (ii) We obtain the complement stability for $B \in \mathcal{B}$ from

$$f^{-1}(B^c) = f^{-1}(Y \setminus B) = f^{-1}(Y) \setminus f^{-1}(B) = X \setminus f^{-1}(B) \in \mathcal{A},$$

  therefore $B^c \in \mathcal{B}$.

(iii) We obtain the σ-$\cup$ stability for a sequence $B_1, B_2, \dots \in \mathcal{B}$, since

$$x \in \bigcup_{n \in \mathbb{N}} f^{-1}(B_n) \iff \exists k \in \mathbb{N} : x \in f^{-1}(B_k) \iff x \in f^{-1}\left(\bigcup_{n \in \mathbb{N}} B_n\right).$$

The union on the left-hand side lies in $\mathcal{A}$, since this is a σ-algebra, and therefore $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

(b) Again, we claim that this is also a σ-algebra.

(i) Since $f^{-1}(Y) = X$ and $Y \in \mathcal{B}$, $X \in f^{-1}(\mathcal{B})$ holds.

(ii) Now let $A \in f^{-1}(\mathcal{B})$ with $A = f^{-1}(B), B \in \mathcal{B}$. Then applies

$$A^c = X \setminus A = f^{-1}(Y) \setminus f^{-1}(B) = f^{-1}(Y \setminus B) \in f^{-1}(\mathcal{B}),$$

since $Y \setminus B = B^c \in \mathcal{B}$.

(iii) Let $A_1 = f^{-1}(B_1), A_2 = f^{-1}(B_2), \dots \in f^{-1}(\mathcal{B})$. Then applies

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} f^{-1}(B_n) = f^{-1}\left(\bigcup_{n \in \mathbb{N}} B_n\right) \in f^{-1}(\mathcal{B}),$$

since $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

Exercise 3 (4 Points)**.**
Let Ω be an uncountable set and

$$\mathcal{A} = \{A \subset \Omega \mid A \text{ or } A^c \text{ is countable}\}.$$

Show that $\mathcal{A}$ is a σ-algebra.

Solution.

(i) The set $\Omega \in \mathcal{A}$. Indeed, by definition, $\emptyset^c = \Omega$ is uncountable (where $\emptyset$ is countable!).

(ii) **Closed under complements**: Let $A \in \mathcal{A}$. Then either A is countable or A^c is countable. If A is countable, then A^c is uncountable, hence $A^c \in \mathcal{A}$. If A^c is countable, then A is uncountable, hence $A \in \mathcal{A}$. In both cases, $A^c \in \mathcal{A}$.

(iii) **Closed under countable unions**: Let $A_1, A_2, A_3, \ldots \in \mathcal{A}$. If any A_n is countable, then $\bigcup_{n=1}^{\infty} A_n$ is countable. If all A_n^c are countable, then $\bigcup_{n=1}^{\infty} A_n^c$ is countable, hence $\bigcup_{n=1}^{\infty} A_n$ is uncountable. In either case, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. Thus, $\mathcal{A}$ is a σ-algebra.

Exercise 4 (4 Points)**.**
Find an example of a Dynkin system that is not a semiring.

Solution.
One could consider $\Omega = \{a,b,c,d\}$ and

$$\mathcal{D} = \{\emptyset, \{a,b\}, \{a,d\}, \{b,c\}, \{c,d\}, \Omega\}.$$

By definition 1.11, $\mathcal{D}$ is a Dynkin system! (You can check this!) However, Definition 1.1(2) is not satisfied in that $\mathcal{D}$ is not closed under $\cap$. Take for instance: $\{a,b\} \cap \{a,d\} = \{a\} \notin \mathcal{D}$.