Measure theory for probabilists   Winter semester 2024

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Tutorial 3 - Further review of toplogy

Exercise 1 (4 Points)**.**
Show that there exists a sequence of open sets $\{O_n\}_{n\in\mathbb{N}} \subset \mathcal{O}$ such that $\bigcap_{n\in\mathbb{N}} O_n$ is not an open set.

Solution.
Consider the cofonite toplogy $\mathcal{O}$ from the previous exercise sheet. Let $O_n = \{1,2,\dots,n\}^c$. Thus, $O_n^c = \{1,2,\dots,n\}$ is finite and

$$\left(\bigcap_{n\in\mathbb{N}} O_n\right)^c = \bigcup_{n\in\mathbb{N}} O_n^c = \bigcup_{n\in\mathbb{N}} \{1,2,\dots,n\} = \mathbb{N},$$

which is not finite. Therefore, the intersection is not an open set.

Exercise 2 (4 Points)**.**
Let $x \in \mathbb{R}^n$, $\epsilon > 0$ and $y \in \mathcal{B}_\epsilon(x)$. Show that

$$\mathcal{B}_{\epsilon-||x-y||}(y) \subseteq \mathcal{B}_\epsilon(x).$$

Solution.
Let $z \in \mathcal{B}_{\epsilon-||x-y||}(y)$, so $||z-y|| < \epsilon - ||x-y|| \implies ||z-y|| + ||x-y|| < r$. Using the triangular inequality, we can write:

$$||x-z|| \leq ||x-y|| + ||z-y|| < r.$$

In this manner, then $z \in \mathcal{B}_\epsilon(x)$ and $\mathcal{B}_{\epsilon-||x-y||}(y) \subseteq \mathcal{B}_\epsilon(x)$.

Exercise 3 (4 Points)**.**
Let $x,y \in \mathbb{R}^n$ and $r = ||x-y||$. Show that

$$\mathcal{B}_{\frac{r}{2}}\left(\frac{x+y}{2}\right) \subseteq \mathcal{B}_r(x) \cap \mathcal{B}_r(y).$$

Solution.
Let $k = \frac{x+y}{2}$ and suppose $z \in \mathcal{B}_{\frac{r}{2}}(k)$. Again, by the triangular inequality:

$$||x-z|| \leq ||x-k|| + ||k-z|| = \frac{1}{2}(||x-y||) + ||m-z|| < \frac{r}{2} + \frac{r}{2} = r$$
$$||y-z|| \leq ||y-k|| + ||k-y|| = \frac{1}{2}(||y-x||) + ||m-z|| < \frac{r}{2} + \frac{r}{2} = r$$

Therefore, $z \in \mathcal{B}_r(x)$, $z \in \mathcal{B}_r(y)$ and $z \in \mathcal{B}_r(x) \cap \mathcal{B}_r(y)$. Hence, $\mathcal{B}_{\frac{r}{2}}\left(\frac{x+y}{2}\right) \subseteq \mathcal{B}_r(x) \cap \mathcal{B}_r(y)$

Exercise 4 (4 Points)**.**

Let (X,r) and (Y,r') be metric spaces and $f : X \to Y$. Show that f is continuous on X if and only if for every closed set A in Y, $f^{-1}(A)$ is closed in X. See Defintion A.1(10) for the general definition on topological spaces.

Solution.

First we shall establish that $f^{-1}(B^c) = (f^{-1}(B))^c$. Take $x \in f^{-1}(B^c)$. So there exists $y \in B^c$ such that $f(x) = y$. Now, suppose that $x \in f^{-1}(B)$. This implies there exist $z \in B$ such that $y = f(x) = z$, which is ridiculous. So, $x \in (f^{-1}(B))^c$. Now, take $x \in (f^{-1}(B))^c$. Therefore, $\forall y \in B, f(x) \neq y$. In this case, $f(x) \in B^c \implies x \in f^{-1}(B^c)$. So we have shown the equality.

Now for a continuous map f and a closed set A, we shall establish that $f^{-1}(A)$ is closed. Well, $(f^{-1}(A))^c = f^{-1}(A^c)$ is open, because A^c is open, by the continuity. We conclude that $f^{-1}(A)$ is closed. Suppose now that for every closed set A,we have $f^{-1}(A)$ being closed. We will use the fact that A is open if A^c is closed. This is true since $((A^c))^c = A$. So, take an open set A. $(f^{-1}(A))^c = f^{-1}(A^c)$ is closed, because A^c is open. Thus, $f^{-1}(A)$ is open and we have proved the continuity of f. In essence, a map is continuous if and only if the preimage of closed sets are closed sets.