universität freiburg

Measure theory for probabilists

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https://pfaffelh.github.io/hp/2024WS_measure_theory.html
https://www.stochastik.uni-freiburg.de/

Tutorial 2 - Review of topology and compactness

Exercise 1.
Let $X = \{a,b,c,d\}$. Which of the following are topologies for X?

(i) $\{\emptyset,X,\{a\},\{b\},\{a,c\},\{a,b,c\},\{a,b\}\}$

(ii) $\{\emptyset,X,\{a\},\{b\},\{a,b\},\{b,d\}\}$

(iii) $\{\emptyset,X,\{a,c,d\},\{b,c,d\}\}$

Can you further give an example of two subsets A and B of $\mathbb{R}$ such that

$$A \cap B = \emptyset, \quad \overline{A} \cap B \neq \emptyset, \quad A \cap \overline{B} \neq \emptyset.$$

Solution.

(i) Yes. Easy to check!

(ii) No! $\{a\} \cup \{b,d\} = \{a,b,d\}$ is in fact not included in the set.

(iii) No! $\{a,c,d\} \cap \{b,c,d\} = \{c,d\}$ is not included in the set.

It holds that $\overline{\mathbb{Q}} = \overline{\mathbb{R}\backslash\mathbb{Q}} = \mathbb{R}$. Thus, $A = \mathbb{Q}$ and $B = \mathbb{R}\backslash\mathbb{Q}$ works. Another example is $A = [0,1) \cup [2,3)$ and $B = [1,2)$.

Exercise 2.
Prove that a subset of a topological space $(\Omega,\mathcal{O})$ is open if and only if its comp0lement in Ω is closed.

Solution. Clear! Let $O \subseteq \Omega$. By definition, $\Omega\backslash O$ is closed if and only if $O = \Omega\backslash(\Omega\backslash O)$ is open.

Exercise 3 (4 Points)**.**
Let A and B be compact subsets of a metric space (X,r). Show that $A \cap B$ and $A \cup B$ are also compact.

Solution.
It is clear that A and B are closed (See Lemma A.8). Furthermore, $A \cap B$ is closed because $A^c \cup B^c = (A \cap B)^c$ which is open. With Corollary A.10, we are done. (Closed subsets of compact sets are compact!). On the other hand, suppose $A \cup B \subseteq \bigcup_{i \in I} O_i$. That is, $O_i, i \in I$ is an open cover of $A \cup B$. Since A and B are compact, there exists finite $J_A \subset I$ and $J_B \subset I$ such that $A \subseteq \bigcup_{i \in J_A} O_i$ and $B \subseteq \bigcup_{i \in J_B} O_i$. Clearly, $J_A \cup J_B \subseteq I$ and $A \cup B \subseteq \bigcup_{i \in J_A \cup J_B} O_i$.

Exercise 4 (4 Points)**.**
Consider the cofinite topology $\mathcal{O}$ on $\mathbb{Z}$ defined as follows: a subset $O \subset \mathbb{Z}$ is an open set if and only if $O = \emptyset$ or O^c is finite. Show that $\mathcal{O}$ is a topology on $\mathbb{Z}$.

Solution. We shall verify the three axioms in Definition A.1(3):

(i) By defintion, $\emptyset$ is an open set so $\emptyset \in \mathcal{O}$ and $\mathbb{Z}$ is open since $\mathbb{Z}^c$ is finite.

(ii) Let O_1,$O_2 \in \mathcal{O}$. We consider two cases: if either O_1 or O_2 is empty, then $O_1 \cap O_2 = \emptyset$, which is in $\mathcal{O}$. If both O_1 and O_2 are non-empty, then both are complements of finite sets. Therefore, we have O_1^c and O_2^c are finite. Thus, $(O_1 \cap O_2)^c = O_1^c \cup O_2^c$, which is a union of finite sets and hence is finite

(iii) Let $\{O_i\}_{i \in I} \subseteq \mathcal{O}$ be an arbitrary collection of open sets. The union of the O_i will only be empty if all O_i are empty. Suppose that there exists some $i \in I$ such that O_i is nonempty, $\left(\bigcap_{i \in I} O_i\right)^c = \bigcup_{i \in I} O_i^c$. If $O_i = \emptyset$ for some $i \in I$, $O = \emptyset$ because of the intersection. Again, we consider two cases: If any O_i is $\emptyset$, then the union $\bigcup_{i \in I} O_i$ may or may not be $\emptyset$. If all O_i are non-empty, then each O_i is the complement of a finite set. The complement of the union is $\left(\bigcup_{i \in I} O_i\right)^c = \bigcap_{i \in I} O_i^c$; if we take the union of an infinite number of non-empty open sets, it will still be open in the cofonite topology. In particular $\mathbb{Z} \backslash (O_1 \cup O_2 \cup O_3 \cup \ldots) = \left(\bigcup_{i \in I} O_i\right)^c$.