universitätfreiburg

Measure theory for probabilists **Winter semester 2024**

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https://pfaffelh.github.io/hp/2024WS_measure_theory.html
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Tutorial 1 - Review of metric spaces

Exercise 1 (4 Points)**.**
If X is a set and $r : X \times X \to \mathbb{R}_+$ is defined by

$$r(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Show that r is a metric on X.
Note: r is in fact called the discrete metric on X.

Solution.
From Definition A.1 (1), we see clearly that conditions (i) and (ii) are satisfied. To show that the triangle inequality holds (condition (iii)), we consider the possible cases for $x,y,z \in X$ and establish that

$$r(x,z) \leq r(x,y) + r(y,z).$$

Case 1: $x = y$ and $y = z$:

$$r(x,z) = 0, \quad r(x,y) = 0, \quad r(y,z) = 0 \implies 0 \leq 0 + 0.$$

Case 2: $x = y$ and $y \neq z$:

$$r(x,z) = 1, \quad r(x,y) = 0, \quad r(y,z) = 1 \implies 1 \leq 0 + 1.$$

Case 3: $x \neq y$ and $y = z$:

$$r(x,z) = 1, \quad r(x,y) = 1, \quad r(y,z) = 0 \implies 1 \leq 1 + 0.$$

Case 4 $x \neq y$ and $y \neq z$:

$$r(x,z) = 1, \quad r(x,y) = 1, \quad r(y,z) = 1 \implies 1 \leq 1 + 1.$$

Thus, r is a metric on X.

Exercise 2 (4 Points)**.**
Show that every mapping from a metric space (Ω,r) to a metric space (Ω',r') is continuous if r is the discrete metric.

Solution.
We first need to recall the definition of continuity in the context of metric spaces. Let (Ω,r) and (Ω',r') be given metric spaces. A function $f : \Omega \to \Omega'$ is continuous at a point $x \in \Omega$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $y \in \Omega$:

$$r(x,y) < \delta \implies r'(f(x),f(y)) < \epsilon.$$

Let $\epsilon > 0$ be given. Since we are working with the discrete metric, we can choose $\delta = 1$. For any $y \in \Omega$,consider the following:
If $y = x$, then $r(x,y) = 0 < \delta$, and we need to show $r'(f(x),f(y)) < \epsilon$. This is trivially true since $r'(f(x),f(x)) = 0 < \epsilon$.
If $y \neq x$, then $r(x,y) = 1$, which is not less than δ. Thus, we do not need to check the condition $r'(f(x),f(y)) < \epsilon$ because the premise $r(x,y) < \delta$ is false. Since the condition $r(x,y) < \delta$ only holds for $y = x$, and at this point $r'(f(x),f(y)) < \epsilon$ holds trivially, we conclude that f is continuous at x. Since x was arbitrary, f is continuous at every point in Ω.

Exercise 3 (4 Points)**.**
Suppose that there is a continuous, one-to-one mapping from a metric space (Ω,r) to a metric space (Ω',r') where r' is the discrete metric. Show that every subset of Ω is open.

Solution.
In the discrete topology, every subset of Ω' is open and a function $f : \Omega \to \Omega'$ is continuous if for every open set $V \subseteq \Omega'$, the preimage $f^{-1}(V)$ is open in Ω. See Definition A.1 (10). Let $A \subseteq \Omega$ be any subset of Ω. We want to show that A is open in Ω. Consider the image of A under f: Since f is continuous and one-to-one, we can consider the image $f(A) \subseteq \Omega'$ and since r' is the discrete metric, every subset of Ω' is open. Therefore, $f(A)$ is an open set in (Ω',r'). Also, because f is continuous, the preimage $f^{-1}(f(A))$ must also be open in Ω. Since f is one-to-one, $f^{-1}(f(A)) = A$. Thus, A is open in Ω. Since A was an arbitrary subset of Ω, we are done.

Exercise 4 (4 Points)**.**
Let X be a complete metric space and let Y be a subset of X. Show that Y is complete if and only if it is closed.

Solution.
Suppose that Y is not complete. Then there is a Cauchy sequence $\{x_n\}_{n\in\mathbf{N}} \subseteq Y$ that does not admit a limit in Y. Since X is complete, $\{x_n\}_{n\in\mathbf{N}}$ converges to some $x \in X$. Thus $x \in X\backslash Y$ (since $x \notin Y$) and $x \in \overline{Y}$, which means that Y is not closed in X. Conversely, suppose Y is complete. Then every Cauchy sequence admits limit in Y. However, each converging sequence in a metric space is a Cauchy sequence, thus $Y = \overline{Y}$.