Measure Theory for Probabilists
17. Projective limits

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Purpose

- Let $X_1, X_2, \ldots$ be coin tosses, i.e. random variables with values in $\{0, 1\}$. What is the joint distribution of $(X_1, X_2, \ldots)$?
- Let $(X_t)_{t \in [0,\infty)}$ some random process. What is its distribution?

- $\rightarrow$ We need to consider probability measures on (uncountably) infinite product spaces!!
- We will do this using our usual construction with outer measures based on a projective family.
- Recall für $H \subseteq J$ the projection $\pi_H^J : \Omega^J \rightarrow \Omega^H$.

Projective family and limit

- $(\Omega, \mathcal{F})$ measurable space, I arbitrary.
- Definition 5.21: A family $(\mathrm{P}_J)_{J \subseteq_f I}$, where P_J is a probability measure on $\mathcal{F}^J := \mathcal{F}^{\otimes J}$, is called projective if

$$\mathrm{P}_H = (\pi^J_H)_* \mathrm{P}_J, \qquad H \subseteq J \subseteq_f I.$$

If there exists a measure P_I on $\mathcal{F}^I := \mathcal{F}^{\otimes I}$ with

$$\mathrm{P}_J = (\pi_J)_* \mathrm{P}_I, \qquad J \subseteq_f I,$$

then we call P_I its projective limit and write

$$\mathrm{P}_I = \varprojlim_{J \subseteq_f I} \mathrm{P}_J.$$

Uniqueness

- Remark 5.23: Projective limits are unique:
  Indeed:

$$\mathcal{H}' := \Big\{ \bigtimes_{i\in J} A_i \times \bigtimes_{i\in I\setminus J} \Omega_i, A_i \in \mathcal{F}_i, i \in J \subseteq_f I \Big\},$$

  is a $\cap$-stable generator of $\mathcal{F}^{\otimes I}$. If $\mathrm{P}_I = \varprojlim_{J\subseteq_f I} \mathrm{P}_J$. and $A = \bigtimes_{i\in J} A_i \times \bigtimes_{i\in I\setminus J} \Omega \in \mathcal{H}'$,

$$\mathrm{P}_I(A) = \mathrm{P}_J\Big(\bigtimes_{i\in J} A_i \Big).$$

Existence

- Theorem 5.24: Let Ω be Polish and $(\mathrm{P}_J)_{J \subseteq_f I}$ a projective family. Then, the projective limit $\varprojlim_{J \subseteq_f I} \mathrm{P}_J$ exists.
- Proof: $\mathcal{H}'$ semi-ring as above. For $A = \times_{i \in J} A_i \times \times_{i \in I \setminus J} \Omega \in \mathcal{H}'$, define

$$\mu(A) := \mathrm{P}_J(\times_{i \in J} A_i)$$

and use the compact system

$$\mathcal{K} := \{\times_{j \in J} K_j \times \times_{i \in I \setminus J} \Omega : J \subseteq_f I, K_j \text{ compact}\} \subseteq \mathcal{H}.$$

To show: μ is inner regular with respect to $\mathcal{K}$.
Then. According to Theorem 2.10, μ is σ-additive.
Furthermore, $\mu(\Omega^I) = 1$, so μ can be uniquely extended to a measure P on $\sigma(\mathcal{H}) = \mathcal{F}^I$ according to Theorem 2.16.

Existence

- Theorem 5.24: Let Ω be Polish and $(\mathrm{P}_J)_{J\subseteq_f I}$ a projective family. Then, the projective limit $\varprojlim_{J\subseteq_f I} \mathrm{P}_J$ exists.
- To show: μ is inner regular with respect to $\mathcal{K}$.
  For $\varepsilon > 0$ and $j \in J$, there is $K_j \subseteq A_j$ cp with $\mathrm{P}_j(A_j \setminus K_j) < \varepsilon$.
  Then,

$$
\begin{aligned}
&\mu\Big(\Big(\bigtimes_{i\in J} A_i \times \bigtimes_{i\in I\setminus J} \Omega\Big) \setminus \Big(\bigtimes_{i\in J} K_i \times \bigtimes_{i\in I\setminus J} \Omega\Big)\Big)\\
&\quad= \mu\Big(\Big((\bigtimes_{i\in J} A_i) \setminus (\bigtimes_{i\in J} K_i)\Big) \times \bigtimes_{i\in I\setminus J} \Omega\Big)\\
&\quad= \mathrm{P}_J\big(\big(\bigtimes_{j\in J} A_j\big) \setminus \big(\bigtimes_{j\in J} K_j\big)\big)\\
&\quad\leq \mathrm{P}_J\Big(\bigcup_{j\in J}(A_j \setminus K_j) \times \bigtimes_{i\neq j} \Omega\Big)\\
&\quad\leq \sum_{j\in J} \mathrm{P}_J\Big((A_j \setminus K_j) \times \bigtimes_{i\neq j} \Omega\Big) = \sum_{j\in J} \mathrm{P}_j(A_j \setminus K_j) \leq |J|\varepsilon.
\end{aligned}
$$