# Measure Theory for Probabilists
# 15. Set systems on product spaces

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# Product spaces

- For an index set $I$ and a family of sets $(\Omega_i)_{i\in I}$, define the product space

$$\Omega := \bigtimes_{i\in I} \Omega_i := \{(\omega_i)_{i\in I} : \omega_i \in \Omega_i\}$$

For $H \subseteq J \subseteq I$, define projections

$$\pi_H^J : \bigtimes_{i\in J} \Omega_i \to \bigtimes_{i\in H} \Omega_i,$$

and $\pi_H := \pi_H^I$ and $\pi_i := \pi_{\{i\}}$, $i \in I$.

# Topology on product spaces

- Definition 5.1: Let $(\Omega_i, \mathcal{O}_i)_{i \in I}$ be a family of topologcal spaces. Then,

$$\mathcal{O} := \mathcal{O}(\mathcal{C}), \qquad \mathcal{C} := \left\{ A_i \times \bigtimes_{j \in I, j \neq i} \Omega_j ; i \in I, A_i \in \mathcal{O}_i \right\}$$

  is called the *product topology* on $\Omega$.
- All $\pi_i, i \in I$ are continuous with respect to the product topology.
  Indeed, for $A_i \in \mathcal{O}_i$,

$$\pi_i^{-1}(A_i) = A_i \times \bigtimes_{I \ni j \neq i} \Omega_j \in \mathcal{C} \subseteq \mathcal{O}.$$

# The product $\sigma$-algebra

- Definition 5.3: Let $(\Omega_i, \mathcal{F}_i)_{i \in I}$ be a family of measurable spaces. Then,

$$\bigotimes_{i \in I} \mathcal{F}_i := \sigma(\mathcal{E}), \qquad \mathcal{E} := \left\{ A_i \times \bigtimes_{j \in I, j \neq i} \Omega_j : i \in I, A_i \in \mathcal{F}_i \right\}$$

  is the *product-$\sigma$-algebra* on $\Omega$.
  We denote the Borel $\sigma$-algebra of $\mathcal{O}$ by $\mathcal{B}(\Omega)$.
- Projections are measurable.
- Lemma 5.5: Let $\mathcal{F}_i = \mathcal{B}(\Omega_i)$. For arbitrary $I$, we have $\bigotimes_{i \in I} \mathcal{B}(\Omega_i) \subseteq \mathcal{B}(\Omega)$. If $I$ is countable and $(\Omega_i, \mathcal{O}_i)_{i \in I}$ are separable metric spaces, then $\mathcal{B}(\Omega) = \bigotimes_{i \in I} \mathcal{B}(\Omega_i)$.
- Proof: Clearly, $\mathcal{C} \subseteq \mathcal{O}(\mathcal{C})$, $\mathcal{C} \subseteq \mathcal{E}$ and $\mathcal{E} \subseteq \sigma(\mathcal{C})$. So,

$$\bigotimes_{i \in I} \mathcal{B}(\Omega_i) = \sigma(\mathcal{E}) = \sigma(\mathcal{C}) \subseteq \sigma(\mathcal{O}(\mathcal{C})) = \mathcal{B}(\Omega).$$

  If $I$ is countable and all spaces are separable, every $A \in \mathcal{O}(\mathcal{C})$ is a countable union of sets in $\mathcal{C}$, so $\mathcal{O}(\mathcal{C}) \subseteq \sigma(\mathcal{C})$. Hence,

$$\sigma(\mathcal{O}(\mathcal{C})) \subseteq \sigma(\sigma(\mathcal{C})) = \sigma(\mathcal{C}).$$

# Products of generators

- Lemma 5.7: Let $(\Omega_i, \mathcal{F}_i)$ be measurable spaces and $\Omega = \times_{i\in I} \Omega_i$.
  1. $I$ finite, $\mathcal{H}_i$ semi-ring with $\sigma(\mathcal{H}_i) = \mathcal{F}_i$. Then
  $$\mathcal{H} := \Big\{ \bigtimes_{i\in I} A_i : A_i \in \mathcal{H}_i, i \in I \Big\}$$
  is semi-ring with $\sigma(\mathcal{H}) = \bigotimes_{i\in I} \mathcal{F}_i$.
  2. $I$ arbitrary, $\mathcal{H}_i$ a $\cap$-stable generator of $\mathcal{F}_i$, $i \in I$. Then
  $$\mathcal{H} := \{ \bigtimes_{i\in J} A_i \times \bigtimes_{i\in I\setminus J} \Omega_i : J \subseteq_f I, A_i \in \mathcal{H}_i, i \in J \}$$
  is $\cap$-stable generator of $\bigotimes_{i\in I} \mathcal{F}_i$.

# $\sigma$-algebra on $\mathbb{R}^d$

- Corollary 5.8: Let $\Omega = \mathbb{R}^d$. For $\underline{a}, \underline{b} \in \mathbb{R}^d$, denote

  $$(\underline{a}, \underline{b}] = (a_1, b_1] \times \cdots \times (a_d, b_d].$$

  Then,

  $$\mathcal{H} := \{(\underline{a}, \underline{b}] : \underline{a}, \underline{b} \in \mathbb{Q}, \underline{a} \leq \underline{b}\}$$

  is a semi-ring with $\sigma(\mathcal{H}) = \mathcal{B}(\mathbb{R}^d)$.
- Proof: $\mathcal{H}$ is a semi-ring that generates $\bigotimes_{i=1}^{d} \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^d)$