

The background of the slide features a large, faint watermark of the University of Bonn seal. The seal is circular and contains a central figure of a seated woman, likely the personification of Wisdom, holding a book. Above her are three portraits of figures, and below her are two more figures. The seal is surrounded by Latin text: 'SIGILLUM UNIVERSITATIS BONONIENSIS' at the top and 'MDCCCXXXIII' at the bottom.

# Measure Theory for Probabilists

## 13. The space $\mathcal{L}^2$

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# A scalar product

- ▶ Apparently,  $\langle \cdot, \cdot \rangle : \mathcal{L}^2 \times \mathcal{L}^2 \rightarrow \mathbb{R}$ , given by

$$\langle f, g \rangle := \mu[fg],$$

is bi-linear, symmetric and positive semi-definite.

- ▶ Complete normed spaces with a scalar product are called Hilbert spaces. So,  $\mathcal{L}^2$  is a Hilbert space.
- ▶ Write  $f \perp g$  iff  $\mu[fg] = 0$

# Parallelogram identity

- ▶ Lemma 4.9: For  $f, g \in \mathcal{L}^2$ ,

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2.$$

- ▶ Proof:

$$\begin{aligned}\|f + g\|^2 + \|f - g\|^2 &= \langle f + g, f + g \rangle + \langle f - g, f - g \rangle \\ &= 2\langle f, f \rangle + 2\langle g, g \rangle = 2\|f\|^2 + 2\|g\|^2.\end{aligned}$$

## Decomposition

- ▶ Proposition 4.10:  $M$  closed, linear subspace of  $\mathcal{L}^2$  and  $f \in \mathcal{L}^2$ . Then, there is an almost everywhere unique decomposition  $f = g + h$  with  $g \in M, h \perp M$ .
- ▶ Proof: For  $f \in \mathcal{L}^2$ , define  $d_f := \inf_{g \in M} \{ \|f - g\| \}$ . Choose  $g_1, g_2, \dots$  with  $\|f - g_n\| \xrightarrow{n \rightarrow \infty} d_f$ . Then

$$\begin{aligned} 4d_f^2 + \|g_m - g_n\|^2 &\leq \|2f - g_m - g_n\|^2 + \|g_m - g_n\|^2 \\ &= 2\|f - g_m\|^2 + 2\|f - g_n\|^2 \xrightarrow{m, n \rightarrow \infty} 4d_f^2. \end{aligned}$$

Thus  $\|g_m - g_n\| \xrightarrow{m, n \rightarrow \infty} 0$ , i.e.  $\|g_n - g\| \xrightarrow{n \rightarrow \infty} 0$  for some  $g \in M$  with  $\|f - g\| = d_f$ . For  $t > 0, l \in M$ ,

$$d_f^2 \leq \|f - g + tl\|^2 = d_f^2 + 2t\langle f - g, l \rangle + t^2\|l\|^2.$$

Since this applies to all  $t$ ,  $\langle f - g, l \rangle = 0$ , i.e.  $f - g \perp M$ .

Uniqueness: Let  $f = g + h = g' + h'$ . Then,  $g - g' \in M$  as well as  $g - g' = h - h' \perp M$ , i.e.  $g - g' \perp g - g'$ . This

means  $\|g - g'\| = \langle g - g', g - g' \rangle = 0$ , i.e.  $g = g'$ .

## Theorem of Riesz-Fréchet

- ▶ Proposition 4.11:  $F : \mathcal{L}^2 \rightarrow \mathbb{R}$  is continuous and linear iff there exists some  $h \in \mathcal{L}^2$  with

$$F(f) = \langle f, h \rangle, \quad f \in \mathcal{L}^2.$$

Then,  $h \in \mathcal{L}^2$  is unique.

- ▶ Proof: ' $\Leftarrow$ ' linearity clear. Continuity:

$$|\langle f - f', h \rangle| \leq \|f - f'\| \cdot \|h\|.$$

For uniqueness, let  $\langle f, h_1 - h_2 \rangle = 0$  for all  $f \in \mathcal{L}^2$ ; in particular, with  $f = h_1 - h_2$

$$\|h_1 - h_2\|^2 = \langle h_1 - h_2, h_1 - h_2 \rangle = 0,$$

thus  $h_1 = h_2$   $\mu$ -almost everywhere.

## Theorem of Riesz-Fréchet

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Then,  $h \in \mathcal{L}^2$  is unique.

- ▶ Proof: ' $\Rightarrow$ ': For  $F = 0$  choose  $h = 0$ . For  $F \neq 0$ ,  $M = F^{-1}\{0\}$  is closed and linear, so for  $f' \in \mathcal{L}^2 \setminus M$ , write  $f' = g' + h'$  with  $g' \in M$  and  $h' \perp M$  and  $F(h') = F(f') - F(g') = F(f') \neq 0$ . Set  $h'' = \frac{h'}{F(h')}$ , so that  $h'' \perp M$  and  $F(h'') = 1$  and for  $f \in \mathcal{L}^2$

$$F(f - F(f)h'') = F(f) - F(f)F(h'') = 0.$$

i.e.  $f - F(f)h'' \in M$ , in particular  $\langle F(f)h'', h'' \rangle = \langle f, h'' \rangle$  and

$$F(f) = \frac{1}{\|h''\|^2} \cdot \langle F(f)h'', h'' \rangle = \frac{1}{\|h''\|^2} \cdot \langle f, h'' \rangle = \langle f, \frac{h''}{\|h''\|^2} \rangle.$$

Now, the assertion follows with  $h := \frac{h''}{\|h''\|^2}$ .