# Measure Theory for Probabilists
# 13. The space $\mathcal{L}^2$

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# A scalar product

- Apparently, $\langle ., . \rangle : \mathcal{L}^2 \times \mathcal{L}^2 \to \mathbb{R}$, given by
$$\langle f, g \rangle := \mu[fg],$$
is bi-linear, symmetric and positive semi-definite.
- Complete normed spaces with a scalar product are called Hilbert spaces. So, $\mathcal{L}^2$ is a Hilbert space.
- Write $f \perp g$ iff $\mu[fg] = 0$

# Parallelogram identity

- Lemma 4.9: For $f, g \in \mathcal{L}^2$,

$$||f+g||^2 + ||f-g||^2 = 2||f||^2 + 2||g||^2.$$

- Proof:

$$\begin{aligned} ||f+g||^2 + ||f-g||^2 &= \langle f+g, f+g\rangle + \langle f-g, f-g\rangle \\ &= 2\langle f, f\rangle + 2\langle g, g\rangle = 2||f||^2 + 2||g||^2. \end{aligned}$$

# Decomposition

- Proposition 4.10: $M$ closed, linear subspace of $\mathcal{L}^2$ and $f \in \mathcal{L}^2$. Then, there is an almost everywhere unique decomposition $f = g + h$ with $g \in M, h \perp M$.
- Proof: For $f \in \mathcal{L}^2$, define $d_f := \inf_{g \in M}\{||f - g|||\}$. Choose $g_1, g_2, \ldots$ with $||f - g_n|| \xrightarrow{n\to\infty} d_f$. Then

$$
\begin{aligned}
4d_f^2 + ||g_m - g_n||^2 &\leq ||2f - g_m - g_n||^2 + ||g_m - g_n||^2 \\
&= 2||f - g_m||^2 + 2||f - g_n||^2 \xrightarrow{m,n\to\infty} 4d_f^2.
\end{aligned}
$$

Thus $||g_m - g_n||^2 \xrightarrow{m,n\to\infty} 0$, i.e. $||g_n - g|| \xrightarrow{n\to\infty} 0$ for some $g \in M$ with $||f - g|| = d_f$. For $t > 0, l \in M$,

$$d_f^2 \leq ||f - g + tl||^2 = d_f^2 + 2t\langle f - g, l\rangle + t^2||l||^2.$$

Since this applies to all $t$, $\langle f - g, l\rangle = 0$, i.e. $f - g \perp M$. Uniqueness: Let $f = g + h = g' + h'$. Then, $g - g' \in M$ as well as $g - g' = h - h' \perp M$, i.e. $g - g' \perp g - g'$. This means $||g - g'|| = \langle g - g', g - g'\rangle = 0$, i.e. $g = g'$.

# Theorem of Riesz-Fréchet

- Proposition 4.11: $F : \mathcal{L}^2 \to \mathbb{R}$ is continuous and linear iff there exists some $h \in \mathcal{L}^2$ with
$$F(f) = \langle f, h \rangle, \qquad f \in \mathcal{L}^2.$$
Then, $h \in \mathcal{L}^2$ is unique.
- Proof: '$\Leftarrow$' linearity clear. Continuity:
$$|\langle |f - f'|, h \rangle| \leq ||f - f'|| \cdot ||h||.$$
For uniqueness, let $\langle f, h_1 - h_2 \rangle = 0$ for all $f \in \mathcal{L}^2$; in particular, with $f = h_1 - h_2$
$$||h_1 - h_2||^2 = \langle h_1 - h_2, h_1 - h_2 \rangle = 0,$$
thus $h_1 = h_2$ $\mu$-almost everywhere.

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Then, $h \in \mathcal{L}^2$ is unique.
- Proof: '$\Rightarrow$': For $F = 0$ choose $h = 0$. For $F \neq 0$, $M = F^{-1}\{0\}$ is closed and linear, so for $f' \in \mathcal{L}^2 \setminus M$, write $f' = g' + h'$ with $g' \in M$ and $h' \perp M$ and $F(h') = F(f') - F(g') = F(f') \neq 0$. Set $h'' = \frac{h'}{F(h')}$, so that $h'' \perp M$ and $F(h'') = 1$ and for $f \in \mathcal{L}^2$
$$F(f - F(f)h'') = F(f) - F(f)F(h'') = 0.$$
i.e. $f - F(f)h'' \in M$, in particular $\langle F(f)h'', h''\rangle = \langle f, h''\rangle$ and
$$F(f) = \frac{1}{||h''||^2} \cdot \langle F(f)h'', h''\rangle = \frac{1}{||h''||^2} \cdot \langle f, h''\rangle = \langle f, \frac{h''}{||h''||^2}\rangle.$$
Now, the assertion follows with $h := \frac{h''}{||h''||^2}$.