Measure Theory for Probabilists
12. Basics of $\mathcal{L}^p$-spaces

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February 6, 2024

Definition of an $\mathcal{L}^p$-space

- For $0 < p \leq \infty$, set

$$\mathcal{L}^p := \mathcal{L}^p(\mu) := \{f : \Omega \to \overline{\mathbb{R}} \text{ measurable with } ||f||_p < \infty\}$$

for

$$||f||_p := (\mu[|f|^p])^{1/p}, \qquad 0 < p < \infty \tag{1}$$

and

$$||f||_\infty := \inf\{K : \mu(|f| > K) = 0\}.$$

Hölder's inequality

- Proposition 4.2.1: f, g be measurable, $0 < p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then,

$$||fg||_r \leq ||f||_p ||g||_q \qquad \text{(Hölder inequality)}$$

- Proof: $p = \infty$ or $||f||_p = 0$, $||f||_p = \infty$, $||g||_q = 0$ or $||g||_q = \infty$: ok, so assume any other case and define

$$\widetilde{f} := \frac{f}{||f||_p}, \qquad \widetilde{g} = \frac{g}{||g||_q}.$$

To show $||\widetilde{f}\widetilde{g}||_r \leq 1$. Convexity of the exponential function:

$$(xy)^r = \exp\left(\tfrac{r}{p} p \log x + \tfrac{r}{q} q \log y\right) \leq \tfrac{r}{p} x^p + \tfrac{r}{q} y^q,$$

and thus

$$||\widetilde{f}\widetilde{g}||_r^r = \mu[(\widetilde{f}\widetilde{g})^r] \leq \tfrac{r}{p}\mu[\widetilde{f}^p] + \tfrac{r}{q}\mu[\widetilde{g}^q] = 1.$$

Minkowski's inequality

- Proposition 4.2.2: For $1 \leq p \leq \infty$,

$$||f+g||_p \leq ||f||_p + ||g||_p.$$

- Proof: $p=1$, $p=\infty$ clear. Else, let $q = p/(p-1)$ and $r = 1/p + 1/q = 1$, so Hölder's inequality gives

$$\begin{aligned} ||f+g||_p^p &\leq \mu[|f| \cdot |f+g|^{p-1}] + \mu[|g| \cdot |f+g|^{p-1}] \\ &\leq ||f||_p \cdot ||(f+g)^{p-1}||_q + ||g||_p \cdot ||(f+g)^{p-1}||_q \\ &= (||f||_p + ||g||_p) \cdot ||f+g||_p^{p-1}, \end{aligned}$$

since

$$\begin{aligned} ||(f+g)^{p-1}||_q = ||(f+g)^{q(p-1)}||_1^{1/q} &= ||(f+g)^p||_1^{(p-1)/p} \\ &= ||f+g||_p^{p-1}. \end{aligned}$$

Dividing by $||f+g||_p^{p-1}$ gives the result.

$p \mapsto \mathcal{L}^p$ is decreasing

- μ finite, $1 \leq r < q \leq \infty$. Then $\mathcal{L}^q(\mu) \subseteq \mathcal{L}^r(\mu)$.
- Counterexample for μ infinite: λ Lebesgue measure, $f : x \mapsto \frac{1}{x} \cdot 1_{x>1}$. Then $f \in \mathcal{L}^2(\lambda)$, but $f \notin \mathcal{L}^1(\lambda)$.
- Proof: $q = \infty$ clear; otherwise since $||1||_p < \infty$,

$$||f||_r = ||1 \cdot f||_r \leq ||1||_p \cdot ||f||_q < \infty$$

for $\frac{1}{p} = \frac{1}{r} - \frac{1}{q} > 0$

$\mathcal{L}^p$-convergence

- Definition 4.6: $f_1, f_2, \ldots$ in $\mathcal{L}^p(\mu)$ *converges to* $f \in \mathcal{L}^p(\mu)$ iff
$$||f_n - f||_p \xrightarrow{n\to\infty} 0.$$
We write $f_n \xrightarrow{n\to\infty}_{\mathcal{L}^p} f$.
- Proposition 4.7: μ be finite, $1 \le r < q \le \infty$ and $f, f_1, f_2, \dots \in \mathcal{L}^q$. If $f_n \xrightarrow{n\to\infty}_{\mathcal{L}^q} f$, then also $f_n \xrightarrow{n\to\infty}_{\mathcal{L}^r} f$.
- Proof: clear since $||f-g||_r \le ||f-g||_q$.

Completeness of $\mathcal{L}^p$

- Proposition 4.8: $p \geq 1, f_1, f_2, \ldots$ be a Cauchy sequence in $\mathcal{L}^p$. Then there is $f \in \mathcal{L}^p$ with $||f_n - f||_p \xrightarrow{n\to\infty} 0$.
- Proof: $\varepsilon_1, \varepsilon_2, \ldots$ summable. There is n_k for each k with $||f_m - f_n||_p \leq \varepsilon_k$ for all $m, n \geq n_k$. In particular,

$$\sum_{k=1}^{\infty} ||f_{n_{k+1}} - f_{n_k}||_p \leq \sum_{k=1}^{\infty} \varepsilon_k < \infty.$$

Monotone convergence and Minkowski give

$$\Big|\Big| \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}| \Big|\Big|_p \leq \sum_{k=1}^{\infty} ||f_{n_{k+1}} - f_{n_k}||_p < \infty.$$

In particular $\sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}| < \infty$ almost everywhere, i.e. for almost all $\omega \in \Omega$, the sequence $f_{n_1}(\omega), f_{n_2}(\omega), \ldots$ is Cauchy in $\mathbb{R}$, hence converges to some f. Fatou gives

$$||f_n - f||_p \leq \liminf_{k\to\infty} ||f_{n_k} - f_n||_p \leq \sup_{m\geq n} ||f_m - f_n||_p \xrightarrow{n\to\infty} 0,$$

i.e. $f_n \xrightarrow{n\to\infty}_{\mathcal{L}^p} f$.