Measure Theory for Probabilists
10. Defining the integral, and some properties

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Outline

- Goal: For a measure μ, define for *many* functions $f : \Omega \to \mathbb{R}$

$$\mu[f] = \int f d\mu = \int f(\omega)\mu(d\omega).$$

- Initial step: For $f = 1_A$ for some $A \in \mathcal{F}$, define

$$\mu[f] := \mu(A).$$

- Definition 3.10: For $f = \sum_{k=1}^{m} c_k 1_{A_k}$ with $c_1, \ldots, c_m \geq 0, A_1, \ldots, A_m \in \mathcal{F}$, define

$$\mu[f] := \sum_{i=1}^{m} c_i \mu(A_i).$$

- Final step: f measurbale: use approximating sequence of simple functions.

Simple properties

- Lemma 3.12: f, g non-negative, simple functions and $\alpha \geq 0$. Then,

$$\mu[af + bg] = a\mu[f] + b\mu[g], \qquad f \leq g \Rightarrow \mu[f] \leq \mu[g].$$

- If $f = 1_A$ for $A \in \mathcal{F}$, note that f is in general not piecewise continuous. In particular, $\int f(x)dx$ does not exist in the sense of Riemann.

Integral of non-negative measurable functions

- Definition 3.14: $(\Omega, \mathcal{F}, \mu)$ measue space, $f : \Omega \to \overline{\mathbb{R}}_+$ measurable. Define

$$\mu[f] := \int f d\mu := \int f(\omega)\mu(d\omega)$$
$$:= \sup\{\mu[g] : g \text{ simple, non-negative, } g \leq f\}.$$

- Definition 3.17: $f : \Omega \to \overline{\mathbb{R}}$ measurable. Then f is said to be μ-integrable if $\mu[|f|] < \infty$,

$$\mu[f] := \int f(\omega)\mu(d\omega) := \int f d\mu := \mu[f^+] - \mu[f^-].$$

- For $A \in \mathcal{F}$ we also write

$$\mu[f, A] := \int_A f d\mu := \mu[f 1_A].$$

Proposition 3.16

- $f, g, f_1, f_2, \dots : \Omega \to \overline{\mathbb{R}}_+$ measurable. Then,
  1. If $f \leq g$, then $\mu[f] \leq \mu[g]$.
  2. If
  $$f_n \uparrow f, \quad \text{then} \quad \mu[f_n] \uparrow \mu[f].$$
  3. If $a, b \geq 0$, then $\mu[af + bg] = a\mu[f] + b\mu[g]$.
- Proof: 1. clear.

  2. Since $f_1, f_2, \dots \leq f$, $\lim_{n\to\infty} \mu[f_n] = \sup_{n\in\mathbb{N}} \mu[f_n] \leq \mu[f]$.
  For the reverse it suffices to show
  $$\mu[g] \leq \sup_{n\in\mathbb{N}} \mu[f_n]$$
  for all simple functions $g = \sum_{k=1}^m c_k 1_{A_k} \leq f$. Let
  $B_n^\varepsilon := \{f_n \geq (1-\varepsilon)g\}$. Since $f_n \uparrow f$ and $g \leq f$, $\bigcup_{n=1}^\infty B_n^\varepsilon = \Omega$
  $$\mu[f_n] \geq \mu[(1-\varepsilon)g1_{B_n^\varepsilon}] = \sum_{k=1}^m (1-\varepsilon)c_k\mu(A_k \cap B_n^\varepsilon)$$
  $$\xrightarrow{n\to\infty} \sum_{k=1}^m (1-\varepsilon)c_k\mu(A_k) = (1-\varepsilon)\mu[g].$$

Some properties

- Define
$$\mathcal{L}^1(\mu) := \left\{ f : \Omega \to \overline{\mathbb{R}} : \mu[|f|^1] < \infty \right\}.$$
- Let $f, g \in \mathcal{L}^1(\mu)$. Then
  1. The integral is monotone, i.e.
  $$f \le g \text{ almost everywhere} \qquad \Longrightarrow \qquad \mu[f] \le \mu[g].$$
  In particular,
  $$|\mu[f]| \le \mu[|f|].$$
  2. The integral is linear, so if $a, b \in \mathbb{R}$, then $af + bg \in \mathcal{L}^1(\mu)$ and
  $$\mu[af + bg] = a\mu[f] + b\mu[g].$$
  3. $g \in \mathcal{L}^1(f_*\mu)$, then $g \circ f \in \mathcal{L}^1(\mu)$ and
  $$\mu[g \circ f] = f_*\mu[g].$$
- Proof: 4. for simple, non-negative functions g. Note $g \circ f = \sum_{k=1}^m c_k 1_{f \in A'_k}$, hence
$$\mu[g \circ f] = \sum_{k=1}^m c_k \mu(f \in A'_k) = \sum_{k=1}^m c_k f_*\mu(A'_k) = f_*\mu[g].$$

Properties almost everywhere

- $f : \Omega \to \overline{\mathbb{R}}_+$ measurable.
  1. $f = 0$ almost everywhere iff $\mu[f] = 0$.
  2. If $\mu[f] < \infty$, then $f < \infty$ almost everywhere.
- Proof: 1. Let $N := \{f > 0\} \in \mathcal{F}$.
  '$\Rightarrow$': $\mu(N) = 0$, so

$$0 \le \mu[f] = \mu[f, N] = \lim_{n\to\infty} \mu[n \wedge f, N] \le \lim_{n\to\infty} \mu[n, N] = 0.$$

'$\Leftarrow$' Let $N_n := \{f \ge 1/n\}$, so $N_n \uparrow N$ and $nf \ge 1_{N_n}$, i.e.

$$0 = \mu[f] \ge \tfrac{1}{n}\mu(N_n).$$

This means that $\mu(N_n) = 0$ and therefore
$\mu(N) = \mu(\bigcup_{n=1}^\infty N_n) = 0$ by σ-sub-additivity of μ.
2. Let $A := \{f = \infty\}$. Since $f1_{f\ge n} \ge n1_{f\ge n}$,

$$\mu(A) = \mu[1_A] \le \mu[1_{f\ge n}] \le \tfrac{1}{n}\mu[f, 1_{f\ge n}] \le \tfrac{1}{n}\mu[f] \xrightarrow{n\to\infty} 0.$$

Lebesgue and Riemann integral

- $f : \mathbb{R} \to \mathbb{R}$ be a piece-wise constant function, i.e.

$$f(x) = \sum_{j=-\infty}^{\infty} a_j 1_{[x_{j-1}, x_j)}(x)$$

  $f : [a, b] \to \mathbb{R}$ is *Riemann-integrable* if $\lambda[|f|] < \infty$ and there are piece-wise constant functions $f_n^- \le f \le f_n^+$ and $\lambda[f_n^+ - f_n^-] \xrightarrow{n\to\infty} 0$. Then, the Riemann integral and Lebesgue integral then coincide.
- $f : \mathbb{R} \to \mathbb{R}$ is called *Riemann-integrable* if $f 1_K$ is Riemann-integrable for all compact intervals $K \subseteq \mathbb{R}$ and $\lambda[f 1_{[-n,n]}]$ converges.

Riemann integrability

- Proposition 3.23: $f : [0, t] \to \mathbb{R}$ piecewise continuous. Then f is integrable, Riemann-integrable, and

$$\lambda[f] = \lim_{n\to\infty} \sum_{k=1}^{\infty} f(y_{n,k})(x_{n,k} - x_{n,k-1})$$

  for $0 = x_{n,0} \leq ... \leq x_{n,k_n} = t$ with $\max_k |x_{n,k} - x_{n,k-1}| \xrightarrow{n\to\infty} 0$ and any $x_{n,k-1} \leq y_{n,k} \leq x_{n,k}$.
- Proof for continuous f. Choose $\varepsilon_n \downarrow 0$ and $x_{n,0} \leq ... \leq x_{n,k_n}$ such that $K \subseteq [x_{n,0}, x_{n,k_n}]$ and $\max_{x_{n,k-1} \leq y < x_{n,k}} |f(x_{n,k-1}) - f(y)| < \varepsilon_n$. Then, find piecewise constant f_n^+, f_n^- with $f_n^- \leq f \leq f_n^+$ and $||f_n^+ - f_n^-|| \leq \varepsilon_n$. Integrability and Riemann-integrability follows. The formula follows from uniform approximation of the function f.

Lebesgue and Riemann integral

- $f = 1_{[0,1]\cap\mathbb{Q}}$ is not Riemann-integrable.
- $f(t) = \frac{(-1)^{\lceil t\rceil+1}}{\lceil t\rceil}$. Then

$$\lambda[f1_{[0,2n]}] = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$
$$= \sum_{k=1}^{n} \frac{1}{2k-1} - \frac{1}{2k} = \sum_{k=1}^{n} \frac{1}{(2k-1)2k}$$

So, f is Riemann-integrable. However

$$\lambda[|f|] = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

So, $|f|$ is not integrable, hence f is not Lebesgue-integrable.