Measure Theory for Probabilists
9. Approximation of measurable functions

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Image measures

- If $\mathcal{F}'$ is a σ-field on Ω', and $f : \Omega \to \Omega'$. Then,

$$\sigma(f) := \{f^{-1}(A') : A' \in \mathcal{F}'\} \text{ is a } \sigma\text{-field on } \Omega.$$

- Definition 2.23: $(\Omega, \mathcal{F}, \mu)$ measure space, $(\Omega', \mathcal{F}')$ measurable space, $f : \Omega \to \Omega'$ with $\sigma(f) \subseteq \mathcal{F}$. Then,

$$\mathcal{F}' \ni A' \mapsto f_*\mu(A') := \mu(f^{-1}(A')) = \mu(f \in A')$$

  is the *image measure* of f under μ.
  If $\mathbb{P}$ is a probability measure, we call $X_*\mu$ the distribution of X under $\mathbb{P}$.

- Proposition 2.25: $f_*\mu$ is a measure on $\mathcal{F}'$.

Lemma 3.2

- $(\Omega', \mathcal{F}')$ measurable space, $f : \Omega \to \Omega'$, $\mathcal{C}' \subseteq \mathcal{F}'$ with $\sigma(\mathcal{C}') = \mathcal{F}'$. Then $\sigma(f^{-1}(\mathcal{C}')) = f^{-1}(\sigma(\mathcal{C}'))$.
- Proof: '$\subseteq$': $f^{-1}(\sigma(\mathcal{C}'))$ is a σ-algebra. So,

$$\sigma(f^{-1}(\mathcal{C}')) \subseteq \sigma(f^{-1}(\sigma(\mathcal{C}'))) = f^{-1}(\sigma(\mathcal{C}'))$$

'$\supseteq$': define

$$\widetilde{\mathcal{F}}' = \{A' \in \sigma(\mathcal{C}') : f^{-1}(A') \in \sigma(f^{-1}(\mathcal{C}'))\} \subseteq \sigma(\mathcal{C}').$$

Again, $\widetilde{\mathcal{F}}'$ is a σ-algebra and $\mathcal{C}' \subseteq \widetilde{\mathcal{F}}' \subseteq \sigma(\mathcal{C}')$. Thus, $\widetilde{\mathcal{F}}' = \sigma(\mathcal{C}')$. For $A' \in \sigma(\mathcal{C}')$, we find

$$f^{-1}(A') \in \sigma(f^{-1}(\mathcal{C}')),$$

which is equivalent to $f^{-1}(\sigma(\mathcal{C}')) \subseteq \sigma(f^{-1}(\mathcal{C}'))$.

Definition 3.3

- $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$ measurable spaces and $f : \Omega \to \Omega'$.
  1. f is *$\mathcal{F}/\mathcal{F}'$-measurable* if $f^{-1}(\mathcal{F}') \subseteq \mathcal{F}$. We define $\sigma(f) := f^{-1}(\mathcal{F}')$ the σ-algebra generated by f.
  2. If $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space and $X : \Omega \to \Omega'$ measurable, then X is called an *Ω'-valued random variable*. The image measure $X_*\mathrm{P}$ from Definition 2.23 is called the *distribution of* X.
  3. If $(\Omega', \mathcal{F}') = (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$, and f is $\mathcal{F}/\mathcal{F}'$-measurable, we say that f is (Borel-)measurable.
  4. If $f = 1_A$ for $A \subseteq \Omega$, then f is called *indicator function*. If $f = \sum_{k=1}^{n} c_k 1_{A_k}$ for $c_1, \dots, c_n \in \overline{\mathbb{R}}$ pairwise different and $A_1, \dots, A_n \subseteq \Omega$, then f is called *simple*.

Examples

- $f:\omega \mapsto \omega$ is measurable, since $f^{-1}(\mathcal{F}) = \mathcal{F}$.
- $(\Omega, \mathcal{O})$ and $(\Omega'.\mathcal{O}')$ topological spaces, $f:\Omega \to \Omega'$ continuous. Then f is measurable.
  Indeed: Since $f^{-1}(\mathcal{O}') \subseteq \mathcal{O}$. From Lemma 3.2,

$$f^{-1}(\mathcal{B}(\Omega')) = f^{-1}(\sigma(\mathcal{O}')) = \sigma(f^{-1}(\mathcal{O}') \subseteq \sigma(\mathcal{O}) = \mathcal{B}(\Omega).$$

- A function $f:\Omega \to \{0,1\}$ is measurable if and only if $f^{-1}(\{1\}) \in \mathcal{F}$. Then, $\sigma(f) = \{\emptyset, f^{-1}(\{1\}), (f^{-1}(\{1\}))^c, \Omega\}$.
- For a non-measurable set/function, see Example 2.27 in the manuscript.

Examples for random variables

- (E, r) metric space, X an E-valued random variable on some probability space, Y an E-valued random variable on another probability space. If $X_*\mathrm{P} = Y_*\mathrm{Q}$, X and Y are *identically distributed* and we write $X \sim Y$.

- Let $(X_i)_{i\in I}$ family of random variables on a probability space. The distribution of $((X_i)_{i\in I})_*\mathrm{P}$ is called the *joint distribution of* $(X_i)_{i\in I}$.

Lemma 3.6

- If $\mathcal{C}' \subseteq \mathcal{F}'$ with $\mathcal{F}' = \sigma(\mathcal{C}')$, then $f : \Omega \to \Omega'$ is $\mathcal{F}/\mathcal{F}'$-measurable if and only if $f^{-1}(\mathcal{C}') \subseteq \mathcal{F}$.
- If $f : \Omega \to \Omega'$ is measurable and $g : \Omega' \to \Omega''$ is measurable, then $g \circ f : \Omega \to \Omega''$ is measuarble.
- A real-valued function f (i.e. $f : \Omega \to \mathbb{R}$) is measurable (with respect to $\mathcal{F}/\mathcal{B}(\mathbb{R})$) if and only if $\{\omega : f(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{Q}$.
- A simple function $f = \sum_{k=1}^{n} c_k 1_{A_k}$ with pairwise different $c_1, \dots, c_n \in \overline{\mathbb{R}}$ and $A_1, \dots, A_n \subseteq \Omega$ is measurable if and only if $A_1, \dots, A_n \in \mathcal{F}$.
- Proof of 1.:
  $f^{-1}(\mathcal{F}') = f^{-1}(\sigma(\mathcal{C}')) = \sigma(f^{-1}(\mathcal{C}')) \subseteq \sigma(\mathcal{F}) = \mathcal{F}$. This means that f is $\mathcal{F}/\mathcal{F}'$-measurable.

Algebraic structures of measurability

- Lemma 3.7: Let $f, g, f_1, f_2, \ldots$ be measurable. Then, the following are measurable: fg, $af + bg$ for $a, b \in \mathbb{R}$, f/g if $g(\omega) \neq 0$ for all $\omega \in \Omega$,

$$\sup_{n\in\mathbb{N}} f_n, \quad \inf_{n\in\mathbb{N}} f_n, \quad \limsup_{n\to\infty} f_n, \quad \liminf_{n\to\infty} f_n.$$

- In particular, $f^+, f^-, |f|$ are measurable.
- Proof: Consider $\psi(\omega) := (f(\omega), g(\omega))$ measruable. Then, $(x, y) \mapsto ax + by$, $(x, y) \mapsto xy$, $(x, y) \mapsto x/y$ are continuous, hence measurbale.
  2. for measurability of $\sup_{n\in\mathbb{N}} f_n$. Write, for $x \in \mathbb{R}$,

$$\left\{\omega : \sup_{n\in\mathbb{N}} f_n(\omega) \leq x\right\} = \bigcap_{n=1}^{\infty} \underbrace{\left\{\omega : f_n(\omega) \leq x\right\}}_{\in\mathcal{F}} \in \mathcal{F}.$$

Approximation by simple functions

- Theorem 3.9: $f : \Omega \to \overline{\mathbb{R}}_+$ measurable. Then there is $f_1, f_2, \cdots : \Omega \to \overline{\mathbb{R}}$ of simple functions with $f_n \uparrow f$.
- Proof: Write

$$f_n(\omega) = n \wedge 2^{-n}[2^n f(\omega)] \uparrow f$$

by construction. Furthermore, $\omega \mapsto [2^n f(\omega)]$ is measurable according to Lemma 3.6.